Notes on Calderón & Zygmund II

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1 Introduction

Consider the truncated singular integral operator

$$(T_\epsilon f)(x) = \int_{|x-y|>\epsilon} K(x,y) f(y)\, dy,$$

for $\epsilon > 0$ and $x \in \mathbb{R}^d$.

Theorem I: Let $K(x,y) = N(x-y)$, where $N : \mathbb{R}^d \setminus \{0\} \to \mathbb{C}$ is positively homogeneous of degree $-d$ and obeys the following:

- $N \in L^1(S^{d-1})$ and $\int_{S^{d-1}} N\, d\sigma = 0$, where σ denotes the standard uniform measure on S^{d-1}; and
- The even component N_{e} defined by $N_{\mathrm{e}}(\omega) = \frac{1}{2}(N(\omega) + N(-\omega))$ on S^{d-1} obeys $\int_{S^{d-1}} |N_{\mathrm{e}}| \log^+ |N_{\mathrm{e}}|\, d\sigma < \infty$ (*i.e.*, belongs to the class $L \log^+ L$).

Then for $f \in L^p(\mathbb{R}^d)$, $1 < p < \infty$, the truncated singular integrals $T_\epsilon f$ converge to a limit Tf in $L^p(\mathbb{R}^d)$ norm, as $\epsilon > 0$. Moreover, $T_* f = \sup_{\epsilon>0} |T_\epsilon f|$ is an L^p function, and $\|T_* f\|_{L^p(\mathbb{R}^d)} \le C \|f\|_{L^p(\mathbb{R}^d)}$, where $C = C(d,p,N)$.

Theorem II: Let $K : (\mathbb{R}^d \times \mathbb{R}^d)^* \to \mathbb{C}$ be defined by $K(x,y) = N(x, x-y)$, where $N : \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}) \to \mathbb{C}$ is a measurable function that is positively-homogeneous in the second variable (of degree $-d$), and satisfies the following:

- For each $x \in \mathbb{R}^d$, $N(x,\cdot) \in L^1(S^{d-1})$ with mean-value zero; and
- There exists $1 < q < \infty$ such that $\sup_{x\in\mathbb{R}^d} \|N(x,\cdot)\|_{L^q(S^{d-1})} < \infty$.

Then the same conclusions as in Theorem I hold for the associated singular integral operators T_ϵ and T_*, in the index range $q' \le p < \infty$.

Theorem III: Assume the kernel $K : (\mathbb{R}^d \times \mathbb{R}^d)^* \to \mathbb{C}$ is given by $K(x,y) = N(x, x-y)\psi(|x-y|)$, where $\psi : \mathbb{R} \to \mathbb{C}$ is the Fourier-Stieltjes transform of a finite Borel measure, and $N : \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}) \to \mathbb{C}$ is measurable and positively-homogeneous of degree $-d$ in the second variable, and obeys the following:

- For all $x \in \mathbb{R}^d$, $y \in S^{d-1}$, $|N(x,y)| \leq F(y)$, where $F : \mathbb{R}^d \setminus \{0\} \to [0,\infty)$ is a positively-homogeneous function of degree $-d$, in $L^1(S^{d-1})$; and
- Either ψ is even and N is odd in the second argument, or ψ is odd and N is even in the second argument.

Then the conclusion of Theorem I also holds for the associated singular integral operators T_ϵ.

Theorem IV: Let K be given as in Theorem III, but with the first condition on N replaced by the following:

- There exists $1 < q < \infty$ such that $\sup_{x\in\mathbb{R}^d} \|N(x,\cdot)\|_{L^q(S^{d-1})} < \infty$.

Then the conclusions of Theorem III hold for the T_ϵ, provided $q' \leq p < \infty$.

Theorem V: Let $K_\epsilon : (\mathbb{R}^d \times \mathbb{R}^d)^* \to \mathbb{C}$ be given by $K_\epsilon(x,y) = \epsilon^{-d}N(x-y)\psi(|x-y|/\epsilon)$, where $\epsilon > 0$, $N : \mathbb{R}^d \setminus \{0\} \to \mathbb{C}$ is a positively-homogeneous function of degree 0 in $L^1(S^{d-1})$, and $\psi : (0,\infty) \to [0,\infty)$ is a decreasing function with $x \mapsto \psi(|x|)$ in $L^1(\mathbb{R}^d)$. Then if $f \in L^p(\mathbb{R}^d)$ for $1 < p \leq \infty$, and

$$(M_* f)(x) = \sup_{\epsilon>0} \left| \int_{\mathbb{R}^d} K_\epsilon(x,y) f(y)\, dy \right|,$$

we have $M_* f \in L^p(\mathbb{R}^d)$ and $\|M_* f\|_{L^p(\mathbb{R}^d)} \leq C\|f\|_{L^p(\mathbb{R}^d)}$, $C = C(d,p,\psi,N)$.

Theorem VI: Let $K_\epsilon : (\mathbb{R}^d \times \mathbb{R}^d)^* \to \mathbb{C}$ be given by $K_\epsilon(x,y) = \epsilon^{-d}N(x,x-y)\psi(|x-y|/\epsilon)$, where $N : \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}) \to \mathbb{C}$ is a measurable, homogeneous of degree 0 in the second variable, with $\sup_{x\in\mathbb{R}^d} \|N(x,\cdot)\|_{L^q(S^{d-1})} < \infty$ for some $1 < q < \infty$; $\psi : (0,\infty) \to \mathbb{R}$ is again a decreasing function which defines an integrable radial profile on $\mathbb{R}^d$. Then the conclusion of Theorem V still holds, in the parameter range $q' \leq p < \infty$.

2 Remarks

We begin by observing how, for any function $f \in L^p(\mathbb{R}^d)$ with p in the specified ranges in our statements, the integral defining $T_\epsilon f$ will be absolutely-convergent for almost every x.

In the case of Theorem I and Theorem III, we take absolute values and expand in polar coordinates about $x \in \mathbb{R}^d$, to obtain

$$\int_{|x-y|>\epsilon} |K(x,y)||f(y)|\, dy = \int_\epsilon^\infty \int_{S^{d-1}} |K(x,x+r\omega)||f(x+r\omega)| r^{d-1}\, dr\, d\sigma.$$

Then, as K in either Theorem I or Theorem III has a positively-homogeneous majorant (either $|N|$ or F, respectively; and both are in $L^1(S^{d-1})$), we have a bound of

$$\int_{S^{d-1}} \int_\epsilon^\infty F(r\omega)|f(x+r\omega)|r^{d-1}\,dr\,d\sigma$$
$$= \int_{S^{d-1}} \int_\epsilon^\infty \frac{F(\omega)}{r^d}|f(x+r\omega)|r^{d-1}\,dr\,d\sigma = \int_{S^{d-1}} F(\omega)\int_\epsilon^\infty \frac{|f(x+r\omega)|}{r}\,dr\,d\sigma$$

and the inner integral can be bounded using Hölder's inequality with r^{-1} and $p' > 1$ (that is, $p < \infty$) to obtain a bound of

$$\int_{S^{d-1}} F(\omega)\cdot C_{\epsilon,p}\left(\int_\epsilon^\infty |f(x+r\omega)|^p\,dr\right)^{1/p}\,d\sigma,$$

and letting $R > 1$, we take $B = B(0,R)$ and integrate over $x \in B$ to obtain

$$\int_B \int_{|x-y|>\epsilon} |K(x,y)||f(y)|\,dy\,dx$$
$$\leq C_{\epsilon,p}\int_{S^{d-1}} F(\omega)\int_B\left(\int_{\mathbb{R}} |f(x+r\omega)|^p\,dr\right)^{1/p}\,dx\,d\sigma$$

by Tonelli.

To this, we apply Hölder's inequality once more, now in the x-integral. We then have a bound of

$$\leq C_{\epsilon,p}\int_{S^{d-1}} F(\omega)\left(\int_B\int_{\mathbb{R}} |f(x+r\omega)|^p\,dr\,dx\right)^{1/p}|B|^{1/p'}\,d\sigma.$$

For each $\omega \in S^{d-1}$, we proceed by rotating x to write

$$\int_B\int_{\mathbb{R}} |f(x+r\omega)|^p\,dr\,dx = \int_B\int_{\mathbb{R}} |(f\circ O_\omega)(z+re_d)|^p\,dr\,dz,$$

and proceed to separate variables to integrate this as

$$\int_{\mathbb{R}}\int_{|z'|<R}\int_{|z_d|<\sqrt{R^2-|z'|^2}} |(f\circ O_\omega)((z',0)+(z_d+r)e_d)|^p\,dz_d\,dz'\,dr.$$

By Tonelli, we may interchange the r integral and write this as

$$\int_{|z'|<R}\int_{|z_d|<\sqrt{R^2-|z'|^2}}\int_{\mathbb{R}} |(f\circ O_\omega)((z',0)+re_d)|^p\,dr\,dz_d\,dz'.$$

Bounding the limits of integration in the z_d-integral by $\pm R$ and evaluating, noting that the internal integral is now z_d-independent, we have

$$\leq 2R\int_{|z'|<R}\int_{\mathbb{R}} |(f\circ O_\omega)((z',0)+re_d)|^p\,dr\,dz' \leq 2R\int_{\mathbb{R}^d} |(f\circ O_\omega)(z)|^p\,dz,$$

which we know is finite. It follows that the relevant integral converges, and thus we have a bound of
$$\leq C_{\epsilon,p,B}\|F\|_{L^1(S^{d-1})}\|f\|_{L^p(\mathbb{R}^d)}.$$

Next, for integrals of the form of Theorem II or Theorem IV, we argue similarly to obtain
$$\leq C_{\epsilon,p}\int_{S^{d-1}}|N(x,\omega)|\left(\int_{\mathbb{R}}|f(x+r\omega)|^p\,dr\right)^{1/p}\,d\sigma,$$
where $q'\leq p<\infty$ is the integrability index associated to f. Then we have a bound of

$$\leq C_{\epsilon,p}\left(\int_{S^{d-1}}|N(x,\omega)|^q\,d\sigma\right)^{1/q}\left(\int_{S^{d-1}}\left(\int_{\mathbb{R}}|f(x+r\omega)|^p\,dr\right)^{q'/p}\,d\sigma\right)^{1/q'}.$$

We bound the integral over $x\in B$ of this quantity by

$$C_{\epsilon,s}\sup_{x\in\mathbb{R}^d}\|N(x,\cdot)\|_{L^q(S^{d-1})}\int_B\left(\int_{S^{d-1}}\left(\int_{\mathbb{R}}|f(x+r\omega)|^p\,dr\right)^{q'/p}\,d\sigma\right)^{1/q'}\,dx,$$

and we use Hölder's inequality in the x-integral with (q',q) to bound the right-most term by

$$|B|^{1/q}\left(\int_{S^{d-1}}\int_B\left(\int_{\mathbb{R}}|f(x+r\omega)|^p\,dr\right)^{q'/p}\,dx\,d\sigma\right)^{1/q'}.$$

Noting that $0<q'/p\leq 1$, we may apply Hölder's inequality in the x-integral once more (this time with index $1\leq p/q'<\infty$) to obtain

$$|B|^{1/q}\left(\int_{S^{d-1}}\left(\int_B\int_{\mathbb{R}}|f(x+r\omega)|^p\,dr\,dx\right)^{q'/p}|B|^{1-q'/p}\,d\sigma\right)^{1/q'},$$

and we use our estimate above for the inner integral jointly with respect to r and x. As a consequence, this is controlled by

$$\leq|B|^{1/q}|B|^{1/q'-1/p}\left(\int_{S^{d-1}}(2R\|f\|^p_{L^p(\mathbb{R}^d)})^{q'/p}\,d\sigma\right)^{1/q'}.$$

We proceed to obtain a bound of

$$\leq C_{\epsilon,p,B}\sup_{x\in\mathbb{R}^d}\|N(x,\cdot)\|_{L^q(S^{d-1})}\|f\|_{L^p(\mathbb{R}^d)},$$

showing that the integral is a.e.-finite, as before.

3 Proof of III, IV

Proof. We start by recalling the theory of the maximally-truncated Hilbert transform: for any $f \in L^p(\mathbb{R})$, $1 < p < \infty$, we have

$$(H_* f)(x) = \sup_{\epsilon > 0} \left| \int_{|x-y|>\epsilon} \frac{f(y)}{x-y}\, dy \right| = \sup_{\epsilon>0} \left| \int_{|y|>\epsilon} \frac{f(x-y)}{y}\, dy \right|$$

a Borel measurable function (as continuity of $\epsilon \in (0, \infty)$ for each fixed x means that we can restrict the supremum to dyadic rationals). (In fact, due to the continuity of $x \mapsto |\int_{|y|>\epsilon} f(x-y)/y\, dy|$, as can be seen by continuity of translation in $L^p(\mathbb{R})$ for $1 \leq p < \infty$, we see that this is actually lower-semicontinuous.)

We also have that

$$H_* f \lesssim M(Hf) + Mf,$$

by Cotlar's inequality.

As a consequence, for any Borel function $\tilde{\epsilon} : \mathbb{R} \to (0, \infty)$, we have that $(x, \epsilon) \mapsto (x, \tilde{\epsilon}(x))$ is a Borel map, and since the map $(\epsilon, x) \mapsto (H_\epsilon f)(x)$ is separately-continuous, it follows that

$$x \mapsto \int_{|x-y|>\tilde{\epsilon}(x)} \frac{f(y)}{x-y}\, dy$$

is a Borel map into $\mathbb{R}$.

Modulating f, we obtain that

$$x \mapsto e^{-2\pi i z x} \int_{|x-y|>\tilde{\epsilon}(x)} e^{2\pi i z y} \frac{f(y)}{x-y}\, dy.$$

Now, by taking repeated products, sums, differences, and compositions, we can see that the function

$$e^{-2\pi i z x} \chi_{|x-y|>\tilde{\epsilon}(x)} e^{2\pi i z y} \frac{f(y)}{x-y}$$

is a real-valued Borel function of $(x, y, z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. Then for each fixed x, after integrating in y, the resulting function of z is Borel measurable.

We integrate with respect to the finite complex Borel measure μ; for fixed x, we have

$$\int_{\mathbb{R}} e^{-2\pi i z x} \left(\int_{|x-y|>\tilde{\epsilon}(x)} e^{2\pi i z y} \frac{f(y)}{x-y}\, dy \right) d\mu_z,$$

and we note that this is absolutely-product-integrable by taking absolute values and using Tonelli;

$$\int_{\mathbb{R}} \left(\int_{|x-y|>\tilde{\epsilon}(x)} \frac{|f(y)|}{|x-y|}\, dy \right) d\mu_z,$$

and we can use Hölder's inequality to estimate the inner integral in y and get a (x-dependent) finite constant.

So we integrate in z and use Fubini to interchange, to obtain

$$\int_{|x-y|>\tilde{\epsilon}(x)} \frac{\widehat{\mu}(x-y)}{x-y} f(y)\, dy,$$

where $\widehat{\mu} = \psi$ is the Fourier-Stieltjes transform of the measure μ.

We claim that, with suitable choice of $\tilde{\epsilon}$, after taking absolute values, this function of x is comparable (in the sense of $\sim$) pointwise to the maximally-truncated Hilbert transform

$$(H_*^\mu f)(x) = \sup_{\epsilon>0} \left| \int_{|x-y|>\epsilon} \frac{\widehat{\mu}(x-y)}{x-y} f(y)\, dy \right|.$$

Indeed, for any fixed x, by the pointwise-everywhere boundedness of $\widehat{\mu}$ and the L^p control on f, we see that for each fixed x, this is a continuous function of $\epsilon \in (0, \infty)$. As a consequence, we may restrict the supremum over dyadic rationals, and select a linearizing map $\tilde{\epsilon} : \mathbb{R} \to 2^{\mathbb{Z}}$ (and this also shows that the function is measurable).

(Strictly speaking, to use this argument, we need to verify beforehand that $H_*^\mu f$ is finite Lebesgue–almost everywhere, but to see this, it suffices to use the dyadic supremum and take functions $\tilde{\epsilon} : \mathbb{R} \to 2^{\mathbb{Z}}$ with finite range, to capture the supremum over finitely-many dyadic rationals; then obtain uniform L^p bounds and pass to the $\sup_{r\in 2^{\mathbb{Z}}}$ limit using monotone convergence.)

Now, by taking L^p norms in x and using Minkowski's integral inequality, together with the maximal-singular-integral bound, we see that

$$\|H_*^\mu f\|_{L^p(\mathbb{R}^d)} \lesssim \int_{\mathbb{R}} \left\| \int_{|x-y|>\tilde{\epsilon}(x)} \frac{f(y) e^{2\pi i z y}}{x-y}\, dy \right\|_{L_x^p(\mathbb{R})} d|\mu|_z \lesssim_p \|f\|_{L^p(\mathbb{R}^d)} \|\mu\|,$$

for $1 < p < \infty$.

From this, we consider a general $f \in L^p(\mathbb{R}^d)$, $1 < p < \infty$. Let $\omega \in S^{d-1}$, and set the directional singular integral

$$(L_\epsilon f)(x, \omega) = \int_{|t|>\epsilon} \frac{f(x - t\omega)}{t} \widehat{\mu}(t)\, dt.$$

Expanding in polar coordinates about (any choice of) x, we know that for σ-a.e. $\omega \in S^{d-1}$, $t \mapsto f(x - t\omega)$ belongs to $L^p(\mathbb{R})$. Fubini's theorem means we get, for $(\lambda^d \otimes \sigma)$-a.e. pairs $(x, \omega) \in \mathbb{R}^d \times S^{d-1}$, that the integral $(L_\epsilon f)(x, \omega)$ is well-defined for all $\epsilon > 0$.

As a consequence, we set

$$(L_* f)(x,\omega) = \sup_{\epsilon>0} |(L_\epsilon f)(x,\omega)| \chi_S(x,\omega),$$

where S denotes the full-measure non-exceptional set where the directional Hilbert transforms are defined.

We claim that $\chi_{N^c}(x,\omega)(L_\epsilon f)(x,\omega)$ is a measurable function in (x,ω). Indeed, we note that the mapping $(x,\omega,t) \mapsto x - t\omega$ is a Borel measurable function (indeed, continuous); it follows that

$$(x,\omega,t) \mapsto \frac{f(x-t\omega)}{t} \widehat{\mu}(t)$$

as a Borel map from $\mathbb{R}^d \times S^{d-1} \times (\mathbb{R} \setminus \{0\}) \to \mathbb{C}$, and it follows (from Tonelli's theorem) that the $\overline{\mathbb{R}}$-valued mapping of (x,ω) given by

$$(x,\omega) \mapsto \int_{|t|>\epsilon} \frac{|f(x-t\omega)|}{|t|} |\widehat{\mu}(t)| \, dt$$

is Borel measurable. And since we know for a.e. (x,ω) this integral is finite, we can restrict to the full-measure set where this occurs and restore real and imaginary, positive and negative parts (taking linear combinations) to get that the function we originally considered is measurable.

Then, again by continuity in $\epsilon \in (0,\infty)$ for fixed $(x,\omega) \in N^c$, we can see that $L_* f$ on this set is given by a countable supremum over dyadic rationals. It follows that $L_* f$ (suitably corrected, on a null set N) is measurable, too.

For any $(x,\omega) \in N^c$, we notice that $(x - r\omega, \omega) \in N^c$ as well, for any $r \in \mathbb{R}$, by the very definition of the set. In that case, we have

$$\begin{aligned}
(L_* f)(x + r\omega, \omega) &= \sup_{\epsilon>0} \left| \int_{|t|>\epsilon} \frac{f(x + r\omega - t\omega)}{t} \widehat{\mu}(t) \, dt \right| \\
&= \sup_{\epsilon>0} \left| \int_{|r-t|>\epsilon} \frac{f(x + r\omega - (r-t)\omega)}{r-t} \widehat{\mu}(r-t) \, dt \right| \\
&= \sup_{\epsilon>0} \left| \int_{|r-t|>\epsilon} \frac{\widehat{\mu}(r-t)}{r-t} f(x + t\omega) \, dt \right| \\
&= (H_*^\mu(t \mapsto f(x+t\omega)))(r),
\end{aligned}$$

so that for any $1 < p < \infty$, and any $(x,\omega) \in N^c$, we get

$$\int_{\mathbb{R}} |(L_* f)(x + r\omega, \omega)|^p \, dr \lesssim_p \|\mu\|^p \int_{\mathbb{R}} |f(x + t\omega)|^p \, dt.$$

We then see this is actually true for any (x,ω): if $(x,\omega) \in N$, the right-hand side will equal ∞. So this is always true for all such pairs.

Then for each fixed ω, we take O_ω to be an isometry mapping $e_d \mapsto \omega$. We have

$$\int_{\mathbb{R}} |(L_* f)(O_\omega(x + re_d), \omega)|^p \, dr \lesssim_p \|\mu\|^p \int_{\mathbb{R}} |(f \circ O_\omega)(x + te_d)|^p \, dt,$$

and we proceed to integrate this inequality along $x \in \mathbb{R}^{d-1} \times \{0\}$, to recover the d-dimensional Lebesgue measure:

$$\int_{\mathbb{R}^d} |(L_* f)(O_\omega z, \omega)|^p \, dz \lesssim_p \|\mu\|^p \int_{\mathbb{R}^d} |(f \circ O_\omega)(z)|^p \, dz,$$

and we apply rotational invariance to see this bound is independent of $\omega \in S^{d-1}$ and also uniform as ω varies over the sphere.

Consequently, if we set

$$(T_S f)(x) = \int_{S^{d-1}} (L_* f)(x, \omega) F(\omega) \, d\sigma$$

(recalling that $L_* f$ is, after correction, Borel; so restriction to $\{x\} \times S^{d-1} \in \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(S^{d-1}) \cong \mathcal{B}(\mathbb{R}^d \times S^{d-1})$ is meaningful), with F being the majorant from the hypotheses of Theorem III, we have

$$\left| \int_{S^{d-1}} (L_\epsilon f)(x, \omega) N(x, \omega) \, d\sigma \right| \leq (T_S f)(x)$$

for all $\epsilon > 0$ and a.e. x. Additionally, we see that

$$\|T_S f\|_{L^p(\mathbb{R}^d)} \leq \int_{S^{d-1}} \|(L_* f)(x, \omega)\|_{L^p_x(\mathbb{R}^d)} F(\omega) \, d\sigma \lesssim_p \|\mu\| \|f\|_{L^p(\mathbb{R}^d)} \|F\|_{L^1(S^{d-1})},$$

using the uniform boundedness of the $L_* f$ integral for varying ω.

But we can also see, for a.e. $x \in \mathbb{R}^d$, $(x, \omega) \in S^{d-1}$ for a.e. $\omega \in S^{d-1}$; and so

$$\int_{S^{d-1}} (L_\epsilon f)(x, \omega) N(x, \omega) \, d\sigma = \int_{S^{d-1}} \int_{|t| > \epsilon} \frac{f(x - t\omega)}{t} \widehat{\mu}(t) \, dt \, d\sigma.$$

We compare with the value (finite for a.e.-x) of

$$\begin{aligned}
(T_\epsilon f)(x) &= \int_{|x-y|>\epsilon} K(x, y) f(y) \, dy = \int_{|x-y|>\epsilon} N(x, x - y) \psi(|x - y|) f(y) \, dy \\
&= \int_\epsilon^\infty \int_{S^{d-1}} N(x, x - (x - r\omega)) \psi(|x - (x - r\omega)|) f(x - r\omega) r^{d-1} \, dr \, d\sigma \\
&= \int_\epsilon^\infty \int_{S^{d-1}} r^{-d} N(x, \omega) \psi(r) f(x - r\omega) r^{d-1} \, dr \, d\sigma \\
&= \int_{S^{d-1}} \int_\epsilon^\infty N(x, \omega) \frac{f(x - r\omega)}{r} \widehat{\mu}(r) \, dr \, d\sigma,
\end{aligned}$$

and we observe that if N is odd in the second variable, and $\widehat{\mu}$ is even, we can use isometry-invariance of σ to also write this as

$$\begin{aligned}
&\int_{S^{d-1}} \int_{\epsilon}^{\infty} N(x,-\omega) \frac{f(x-r(-\omega))}{r} \widehat{\mu}(r)\, dr\, d\sigma \\
&= \int_{S^{d-1}} \int_{\epsilon}^{\infty} -N(x,\omega) \frac{f(x+r\omega)}{r} \widehat{\mu}(r)\, dr\, d\sigma \\
&= \int_{S^{d-1}} \int_{-\epsilon}^{-\infty} -N(x,\omega) \frac{f(x-r\omega)}{-r} \widehat{\mu}(-r)\, \frac{dr}{-1}\, d\sigma \\
&= \int_{S^{d-1}} \int_{-\infty}^{-\epsilon} N(x,\omega) \frac{f(x-r\omega)}{r} \widehat{\mu}(r)\, dr\, d\sigma,
\end{aligned}$$

so that averaging the two results in

$$(T_\epsilon f)(x) = \frac{1}{2} \int_{S^{d-1}} (L_\epsilon f)(x,\omega) N(x,\omega)\, d\sigma,$$

for a.e. x.

Likewise, if N is even, and $\widehat{\mu}$ is odd, we once again obtain the same result, but this time from $N(x,\omega)\frac{f(x-r\omega)}{-r}(-\widehat{\mu}(r))$ in the integrand.

So, in either case, we can estimate $\|T_* f\|_{L^p(\mathbb{R}^d)} = \|\sup_{\epsilon>0} |T_\epsilon f|\|_{L^p(\mathbb{R}^d)} \leq \frac{1}{2}\|T_S f\|_{L^p(\mathbb{R}^d)} \lesssim_p \|\mu\| \|f\|_{L^p(\mathbb{R}^d)} \|F\|_{L^1(S^{d-1})}$ (using the restriction to dyadic rationals to ensure the pointwise control holds outside a union of countably-many null sets, and using the uniform-in-ϵ bound on the right-hand integral).

This proves Theorem III. For Theorem IV, we modify our arguments by setting

$$(T_S f)(x) = \int_{S^{d-1}} (L_* f)(x,\omega) |N(x,\omega)|\, d\sigma$$

instead, and then we still have

$$|(T_\epsilon f)(x)| = \frac{1}{2} \left| \int_{S^{d-1}} (L_\epsilon f)(x,\omega) N(x,\omega)\, d\sigma \right| \leq \frac{1}{2} (T_S f)(x),$$

and we get

$$(T_S f)(x) \leq \left(\int_{S^{d-1}} |(L_* f)(x,\omega)|^p\, d\sigma \right)^{1/p} \left(\int_{S^{d-1}} |N(x,\omega)|^{p'}\, d\sigma \right)^{1/p'}.$$

For the right-most term here, we use that $p \geq q'$ implies $p' \leq q$, so we may apply Hölder's inequality to control the final term by $\lesssim_{d,p,q} \|N(x,\cdot)\|_{L^p(S^{d-1})}$. Then, taking pth powers and integrating in x, and using the uniform bound, we obtain the same result as before. This shows Theorem IV. $\square$

We conclude by showing that $T_\epsilon f \to Tf$ in $L^p(\mathbb{R}^d)$.

To begin, we take an auxiliary function $\rho \in C_c^\infty(\mathbb{R})$, radially decreasing with $\rho = 1$ in a neighborhood of the origin, and $\operatorname{supp}(\rho) \subseteq (-1, 1)$.

We then consider the function $t \mapsto \widehat{\mu}(t)\rho(t) \in C_c(\mathbb{R})$, and we see that

$$\begin{aligned}\mathcal{F}(\widehat{\mu}\rho)(\xi) &= \int_{\mathbb{R}} e^{-2\pi i \xi t}\rho(t)\widehat{\mu}(t)\,dt = \int_{\mathbb{R}} e^{-2\pi i \xi t}\rho(t)\int_{\mathbb{R}} e^{-2\pi i t z}\,d\mu_z\,dt \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \rho(t)e^{-2\pi i(\xi+z)t}\,dt\,d\mu_z = \int_{\mathbb{R}} \widehat{\rho}(\xi+z)\,d\mu_z.\end{aligned}$$

It is clear that $\mathcal{F}(\widehat{\mu}\rho) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, by Minkowski's integral inequality.

From calculation, we know that the functions $t^{-1}\chi_{\epsilon<|t|<\epsilon^{-1}}$ all belong to $L^2(\mathbb{R})$ for $0 < \epsilon < 1$, and with uniformly-bounded Fourier transforms in ϵ. As a consequence, Plancherel (with dominated convergence) shows that

$$\lim_{\epsilon\downarrow 0}\int_{|t|>\epsilon}\frac{\widehat{\mu}(t)\rho(t)}{t}\,dt = c\int_{\mathbb{R}} \operatorname{sign}(\xi)\mathcal{F}(\widehat{\mu}\rho)(\xi)\,d\xi;$$

this limit exists.

It follows that

$$\int_{|x-y|>\epsilon} K(x,y)\rho(|x-y|)\,dy = \int_\epsilon^\infty \int_{S^{d-1}} r^{-d}N(x,\omega)\widehat{\mu}(r)\rho(r)r^{d-1}\,dr\,d\sigma,$$

or

$$\int_\epsilon^\infty \int_{S^{d-1}} N(x,\omega)\frac{\widehat{\mu}(r)\rho(r)}{r}\,dr\,d\sigma.$$

In the case when N is odd in the second variable, we observe that we can separate the integration in ω and r, to see that this equals the mean-value of N over the sphere: zero.

Otherwise, if N is even, $\widehat{\mu}$ must be odd, by our conditions in Theorems III and IV. Then we observe that by a change-of-variable, this also equals

$$\int_{S^{d-1}}\int_{-\infty}^{-\epsilon}\frac{\widehat{\mu}(-r)\rho(-r)}{-r}\,dr\,d\sigma = \int_{S^{d-1}} N(x,\omega)\int_{-\infty}^{-\epsilon}\frac{\widehat{\mu}(r)\rho(r)}{r}\,dr\,d\sigma.$$

Averaging and using the existence of the principal-value type limit, we see that this pointwise limit exists.

We proceed by taking $f \in C_c^\infty(\mathbb{R}^d)$, and writing

$$\begin{aligned}(T_\epsilon f)(x) = \int_{|x-y|>\epsilon} & K(x,y)(f(y) - f(x)\rho(|x-y|))\,dy \\ & + f(x)\int_{|x-y|>\epsilon} K(x,y)\rho(|x-y|)\,dy.\end{aligned}$$

We see that

$$\lim_{\epsilon\downarrow 0}(T_\epsilon f)(x) = \int_{\mathbb{R}^d} K(x,y)(f(y)-f(x)\rho(|x-y|))\,dy \\ + c_{\mu,\rho} f(x)\int_{S^{d-1}} N(x,\omega)\,d\sigma,$$

where in the first integral, the behavior of the integrand as $O(|x-y|)$ near $y=x$ removes the singularity; the compact support in y ensures that the integral converges. The convergence of the second term was shown, earlier.

This object Tf also lies in $L^p(\mathbb{R}^d)$, by pointwise convergence and Fatou's lemma, along with the control on the maximal function $T_* f$. So, for $f \in C_c^\infty(\mathbb{R}^d)$, we have $\|Tf\|_{L^p(\mathbb{R}^d)} \lesssim_{p,K} \|f\|_{L^p(\mathbb{R}^d)}$. It follows that there exists a unique linear extension of this operator to all of $L^p(\mathbb{R}^d)$.

Since we have shown pointwise convergence for a dense subset ($C_c^\infty(\mathbb{R}^d)$) of $L^p(\mathbb{R}^d)$. Because we have a maximal inequality on $T_* f$, we get pointwise-a.e. convergence for all $f \in L^p(\mathbb{R}^d)$.

Finally, since we have pointwise convergence of the $T_\epsilon f \to Tf$ almost everywhere, and the $T_\epsilon f$ are all uniformly dominated by $T_* f \in L^p(\mathbb{R}^d)$, dominated convergence implies convergence of the $T_\epsilon f$ in $L^p(\mathbb{R}^d)$, also.

4 Proof of V, VI

Proof. We let $f \in L^p(\mathbb{R}^d)$ be a nonnegative function, and we take $\phi(t) = \psi(t)t^{d-1}$, where the profile ψ is as in V, VI. We then have $\phi \in L^1(0,\infty)$.

Now, we define the maximal function via

$$(M_\phi f)(r) = \sup_{\lambda>0}\frac{1}{\lambda}\int_0^\infty |f(r+t)|\phi(t/\lambda)\,dt.$$

It follows from the radial majorant lemma that

$$\|M_\phi f\|_{L^p(\mathbb{R})} \lesssim_p \|f\|_{L^p(\mathbb{R})}\|\phi\|_{L^1(0,\infty)}.$$

Consequently, in the multidimensional setting, for $f \in L^p(\mathbb{R}^d)$ we take

$$(M_\phi f)(x,\omega) = \sup_{\lambda>0}\frac{1}{\lambda}\int_0^\infty |f(x+t\omega)|\phi(t/\lambda)\,dt,$$

and we have

$$\int_{\mathbb{R}} |(M_\phi f)(x+r\omega,\omega)|^p\,dr = \int_{\mathbb{R}} |(M_\phi(t\mapsto f(x+t\omega)))(r)|^p\,dr \\ \lesssim_p \int_{\mathbb{R}} |f(x+t\omega)|^p\,dt.$$

Now, again changing variables, rotating, and integrating over the hyperplane $\{x_d = 0\}$, we get that

$$\int_{\mathbb{R}^d} |(M_\phi f)(z,\omega)|^p \, dz \lesssim_p \int_{\mathbb{R}^d} |f(z)|^p \, dz,$$

uniformly in ω. (To avoid technicalities, we may restrict to f bounded and compactly-supported, and pass to general f by monotone convergence.)

In the case of Theorem V,

$$(M_* f)(x) = \sup_{\epsilon>0} \int_{\mathbb{R}^d} \epsilon^{-d} |N(x-y)| \psi(|x-y|/\epsilon) |f(y)| \, dy,$$

and to avoid technical issues with measurability, we again first postulate f compactly-supported and bounded. The general case will again follow by using positivity and monotonicity, and taking limits of our truncations.

Then, in the case of Theorem V, we have a uniform bound of

$$\leq \frac{1}{\epsilon} \int_{S^{d-1}} \int_0^\infty |N(\omega)| \phi(r/\epsilon) f(x+r\omega) \, dr \, d\sigma \leq \int_{S^{d-1}} |N(\omega)| (M_\phi f)(x,\omega) \, d\sigma$$

in ϵ. Accordingly, we take the $L^p(\mathbb{R}^d)$ norm in x, and once again interchange with the integral sign, to get a bound of

$$\lesssim_p \|N\|_{L^1(S^{d-1})} \|f\|_{L^p(\mathbb{R}^d)}.$$

In the case of Theorem VI, we have

$$(M_* f)(x) \leq \int_{S^{d-1}} |N(x,\omega)| (M_\phi f)(x,\omega) \, d\sigma,$$

and this again follows by using the (p', p) Hölder's inequality and using that $p' \leq q$ on the first term. Raising to the pth power and integrating, we obtain the result with

$$\lesssim_{d,p,q} \sup_{x \in \mathbb{R}^d} \|N(x,\cdot)\|_{L^q(S^{d-1})} \|f\|_{L^p(\mathbb{R}^d)}.$$

$\square$

5 Proof of I, II

The proof of the first two results, in complete generality, is quite involved. In Theorem I, we note that if N is an odd kernel ($N(-\omega) = -N(\omega)$), application of Theorem III with $\psi(t) = 1$ (which is the Fourier-Stieltjes transform of $\mu = \delta_0$; $\int_{\mathbb{R}} e^{-2\pi i t x} \, d\mu_x = e^{-2\pi i t(0)} = 1$) is possible, and we obtain the result.

Likewise, in Theorem II, if $N(x,\cdot)$ is odd, we can ensure that Theorem IV applies. So we may subtract-off the odd component of each kernel, and assume that N in Theorem I and $N(x,\cdot)$ in Theorem II are even functions.

We now recall the vector Riesz transform

$$\vec{R} = (R_1, \ldots, R_d), \qquad R_i f = c_{i,d} \frac{x_i}{|x|^{d+1}} * f,$$

normalized to have Fourier transforms $\widehat{R_i f} = \frac{\xi_i}{|\xi|}$.

As a consequence,

$$\vec{R} \cdot (\vec{R} f) = f,$$

for any scalar-valued function $f \in L^p(\mathbb{R}^d)$ (since $\sum_i \xi_i^2/|\xi|^2 = 1$ on $\mathbb{R}^d \setminus \{0\}$).

We proceed to let $\vec{K}$ denote the vector kernel of the vector Riesz transform. We take any $F : \mathbb{R}^d \setminus \{0\} \to \mathbb{C}$ which is positively homogeneous, of degree $-d$, with F in $L \log^+ L$ on the unit sphere and mean-zero. We let $\phi \in C^\infty(\mathbb{R}^d; [0,1])$ be a radially-increasing function, vanishing in $B(0, \frac{1}{4})$, equal to 1 outside $B(0, \frac{3}{4})$.

Then we define, for $x \neq 0$ and $\epsilon > 0$,

$$F_1^\epsilon(x) = \lim_{\delta \downarrow 0} \int_{|x-y|>\epsilon} \chi_{B(0,\delta)^c}(y) \vec{K}(x-y) F(y)\, dy,$$

$$F_2^\epsilon(x) = \int_{|x-y|>\epsilon} \vec{K}(x-y) F(y) \phi(y)\, dy.$$

Each F_2^ϵ is absolutely convergent everywhere; ϕ eliminates the singularity at the origin for F, the truncation keeps $\vec{K}$ bounded, and for large y, $\vec{K}(x-y)$ behaves as $|y|^{-d}$, which gives a convergent integral when multiplied with $F(y)$.

For F_1^ϵ, we can write

$$F_1^\epsilon(x) = \int_{|x-y|>\epsilon} \chi_{B(0,\delta_0)^c}(y) F(y) \vec{K}(x-y)\, dy \\ + \int_{\delta<|y|<\delta_0} \chi_{|x-y|>\epsilon} F(y) (\vec{K}(x-y) - \vec{K}(x))\, dy,$$

by the mean-zero condition on F, and using the smoothness of the Riesz kernel to obtain an absolutely-convergent integral for the second term. Here, we take $0 < \delta_0 < \frac{1}{4}|x|$. Consequently, these integrals all converge as $\delta \downarrow 0$.

Claim: As $\epsilon \downarrow 0$, F_1^ϵ and F_2^ϵ converge to functions $F_1 : \mathbb{R}^d \setminus \{0\} \to \mathbb{C}^d$, $F_2 : \mathbb{R}^d \setminus \{0\} \to \mathbb{C}^d$, in the spaces $L^1_{\mathrm{loc}}(\mathbb{R}^d \setminus \{0\})$ and $L^1_{\mathrm{loc}}(\mathbb{R}^d)$, respectively. Both F_1 and F_2 are odd, and F_1 is homogeneous of degree $-d$.

Moreover, for $|x| \geq 1$, we have

$$|F_1(x) - F_2(x)| \lesssim \|F\|_{L^1(S^{d-1})} |x|^{-d-1},$$

and for F_2, there exists a homogeneous function $G \in L^1(S^{d-1})$ of degree 0 such that $|F_2(x)| \leq G(x)$ in $|x| \leq 1$.

And, if there exists a $1 < q < \infty$ such that $F \in L^q(S^{d-1})$, then we have $\|F_1\|_{L^q(S^{d-1})} + \|G\|_{L^q(S^{d-1})} \lesssim \|F\|_{L^q(S^{d-1})}$.

Proof. Since $\vec{K}$ is odd while F is even, we see that F_1^ϵ and F_2^ϵ are odd.

We also see that each F_1^ϵ satisfies the desired homogeneity, and if we have convergence in $L^1_{\text{loc}}(\mathbb{R}^d \setminus \{0\})$, we may consider polar coordinates and recover that the limiting function, for almost every direction, is $(-d)$-homogeneous in almost all λ. The limit follows by correcting on a set of measure zero.

To obtain convergence of the F_1^ϵ in $L^1_{\text{loc}}(\mathbb{R}^d \setminus \{0\})$, it suffices to show

$$\lim_{\epsilon,\epsilon' \downarrow 0} \int_{r \le |y| \le R} |F_1^\epsilon(x) - F_1^{\epsilon'}(x)|\, dx = 0$$

for any pair of radii (r, R) with $0 < r < R < \infty$.

For this, we take $\delta_0 \ll \frac{r}{8}$, so that we can write (taking $0 < \epsilon < \epsilon'$, without loss)

$$\begin{aligned} F_1^\epsilon(x) - F_1^{\epsilon'}(x) = &\int_{\epsilon < |x-y| < \epsilon'} \chi_{|y| \ge \delta_0} F(y) \vec{K}(x-y)\, dy \\ &+ \int_{|y| < \delta_0} \chi_{\epsilon < |x-y| < \epsilon'} F(y)(\vec{K}(x-y) - \vec{K}(x))\, dy. \end{aligned}$$

We note that the second term vanishes when $\epsilon, \epsilon' > 0$ are sufficiently small; indeed, $\chi_{\epsilon < |x-y| < \epsilon'}$ will then only be nonzero when y is extremely close to x; but then, $|y| < \delta_0 < \frac{r}{8} < \frac{|x|}{2}$ will be impossible.

So the main task is to control the term

$$\int_{|y| \ge \delta_0} F(y) \chi_{\epsilon < |x-y| < \epsilon'} \vec{K}(x-y)\, dy,$$

and we see this is equal to

$$(\vec{T}_\epsilon(F\chi_{B^c}))(x) - (\vec{T}_{\epsilon'}(F\chi_{B^c}))(x),$$

where $\vec{T}_\epsilon$ is the truncated singular integral operator associated with the vector Riesz transform.

The next step, showing local L^1 convergence away from the origin at $L \log^+ L$ regularities, is unpleasant. First, we note that if $\epsilon, \epsilon' > 0$ are small, and $r \le |x| \le R$, then the only values of y which end up mattering are those within $|x-y| < \max(\epsilon, \epsilon')$, meaning $|y| \le 2R$. So we can limit our consideration to a local annulus for F: that is, we can write this as

$$(\vec{T}_\epsilon(F\chi_A))(x) - (\vec{T}_{\epsilon'}(F\chi_A))(x).$$

Now, to show integrability, we will prove a refined distributional inequality for Calderón-Zygmund operators. Let $\lambda > 0$, and take $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$.

We will decompose

$$f_{<\lambda/8} = f\chi_{\{|f| \le \lambda/8\}}, \qquad f_{\ge \lambda/8} = f - f_{<\lambda/8},$$

and if T is any first-generation Calderón-Zygmund operator, for $0 < \epsilon < 1$, we have the basic inclusion

$$\{|T_\epsilon f| > \lambda\} \subseteq \{|T_\epsilon(f_{<\lambda/8})| > \lambda/2\} \cup \{|T_\epsilon(f_{\geq\lambda/8})| > \lambda/2\}.$$

By Chebyshev's inequality, we estimate the measure of the first set as follows:

$$\lambda^d(\{|T_\epsilon(f_{<\lambda/8})| > \lambda/2\}) \leq \frac{4}{\lambda^2}\int_{\mathbb{R}^d} |T_\epsilon(f_{<\lambda/8})|^2 \, d\lambda^d \lesssim_T \frac{1}{\lambda^2}\int_{|f|<\lambda/8} |f|^2 \, d\lambda^d,$$

using the definition of $f_{<\lambda/8}$ and the fact that the truncations T_ϵ, as Calderón-Zygmund operators, are L^2-bounded (uniformly in $\epsilon > 0$).

For the second set, we will take the Calderón-Zygmund decomposition of the function $f_{\geq\lambda/8}$ (importantly, not f), at height λ. Thus we produce a countable family of dyadic cubes $(Q_n)_n \subseteq \mathcal{D}$ such that we have the good-function

$$g(x) = \begin{cases} f_{\geq\lambda/8}(x) & x \notin \bigcup_n Q_n \\ \frac{1}{\lambda^d(Q_n)}\int_{Q_n} f_{\geq\lambda/8} \, d\lambda^d & x \in Q_n \end{cases}$$

and the bad functions

$$b(x) = \sum_n \chi_{Q_n}\left(f_{\geq\lambda/8} - \frac{1}{\lambda^d(Q_n)}\int_{Q_n} f_{\geq\lambda/8} \, d\lambda^d\right) = \sum_Q b_Q,$$

with $f_{\geq\lambda/8} = g + b$. We make the observations that pointwise,

$$\begin{aligned} |g| &\leq |f_{\geq\lambda/8}|\chi_{(\bigcup_n Q_n)^c} + 2^d\lambda\chi_{\bigcup_n Q_n} \leq |f_{\geq\lambda/8}|\chi_{\{|f_{\geq\lambda/8}|\leq\lambda\}} + 2^d\lambda\chi_{\bigcup_n Q_n} \\ &= |f|\chi_{|f|\geq\lambda/8}\chi_{|f_{\geq\lambda/8}|\leq\lambda} + 2^d\lambda\chi_{\bigcup_n Q_n} = |f|\chi_{\lambda/8\leq|f|\leq\lambda} + 2^d\lambda\chi_{\bigcup_n Q_n} \end{aligned}$$

and in norm,

$$\int_{\mathbb{R}^d} |b_Q| \, d\lambda^d \leq \int_{Q_n} |f_{\geq\lambda/8}| \, d\lambda^d + \left|\int_{Q_n} f_{\geq\lambda/8} \, d\lambda^d\right| \leq 2\int_{Q_n} |f_{\geq\lambda/8}| \, d\lambda^d.$$

As usual, using $f_{\geq\lambda/8} = g + b$, we will split

$$\{|T_\epsilon(f_{\geq\lambda/8})| > \lambda/2\} \subseteq \{|T_\epsilon g| > \lambda/4\} \cup \{|T_\epsilon b| > \lambda/4\}.$$

Repeating the use of Chebyshev's inequality and L^2-boundedness, with our pointwise inequality for g, we can estimate the measure of the former set by

$$\begin{aligned} \lesssim_T \frac{1}{\lambda^2}\int_{\mathbb{R}^d} |g|^2 \, d\lambda^d &\lesssim_d \frac{1}{\lambda^2}\int_{\lambda/8\leq|f|\leq\lambda} |f|^2 \, d\lambda^d + \frac{1}{\lambda^2}\int_{\bigcup_n Q_n} \lambda^2 \, d\lambda^d. \\ &\lesssim \frac{1}{\lambda^2}\int_{\lambda/8\leq|f|\leq\lambda} |f|^2 \, d\lambda^d + \lambda^d\left(\bigcup_n Q_n\right). \end{aligned}$$

Finally, we estimate the bad terms in the usual sense: we take a buffered neighborhood of the cubes (Q_n), and its exterior, summing $|T_\epsilon b| \leq \sum_Q |T_\epsilon b_Q|$ and bounding the excess contribution associated to each cube individually:

$$\{|T_\epsilon b| > \lambda/4\} \subseteq \bigcup_n (2Q_n) \cup \bigcup_Q \{|T_\epsilon b_Q| \chi_{(\bigcup_n (2Q_n))^c} > \lambda/4\},$$

where the measure of the first set is clearly controlled by

$$\leq \sum_n \lambda^d(2Q_n) = 2^d \sum_n \lambda^d(Q_n) = 2^d \lambda^d\Big(\bigcup_n Q_n\Big).$$

For the second set, we use Chebyshev's inequality, along with the representation formula for T_ϵ: we have

$$\leq \sum_Q \frac{4}{\lambda} \int_{\mathbb{R}^d} \chi_{(\bigcup_Q (2Q))^c} |(T_\epsilon b_Q)(x)|\, dx \lesssim \sum_Q \frac{1}{\lambda} \int_{(2Q)^c} |(T_\epsilon b_Q)(x)|\, dx,$$

where each

$$(T_\epsilon b_Q)(x) = \int_{\mathbb{R}^d} (K(x,y) - K(x, c_Q)) b_Q(y)\, dy,$$

for $x \notin 2Q$, and using the mean-zero property. Taking absolute values and integrating in x (interchanging and applying the Hörmander property on the kernel) ensures that we can bound the sum of the $\|T_\epsilon b_Q\|_{L^1(\mathbb{R}^d)}$ contributions as

$$\lesssim \frac{1}{\lambda} \sum_Q \int_{\mathbb{R}^d} |b_Q(y)|\, dy \leq \frac{1}{\lambda} \sum_Q 2 \int_Q |f_{\geq \lambda/8}(y)|\, dy \lesssim \frac{1}{\lambda} \int_{\mathbb{R}^d} |f_{\geq \lambda/8}(y)|\, dy,$$

using our estimate on the $\|b_Q\|_{L^1(\mathbb{R}^d)}$ terms, above.

As a consequence, we get that

$$\begin{aligned}\lambda^d(\{|T_\epsilon f| > \lambda\}) \lesssim_{d,T} \frac{1}{\lambda^2} \int_{|f| < \lambda/8} |f|^2\, d\lambda^d + \frac{1}{\lambda^2} \int_{\lambda/8 \leq |f| \leq \lambda} |f|^2\, d\lambda^d \\ + \lambda^d\Big(\bigcup_n Q_n\Big) + \lambda^d\Big(\bigcup_n Q_n\Big) + \frac{1}{\lambda} \int_{\mathbb{R}^d} |f| \chi_{|f| \geq \lambda/8}\, d\lambda^d.\end{aligned}$$

We recall that the measure of the bad cubes associated to $f_{\geq \lambda/8}$ at height λ is controlled, in the sense that

$$\lambda^d\Big(\bigcup_n Q_n\Big) \simeq \frac{1}{\lambda} \sum_n \int_{Q_n} |f_{\geq \lambda}|\, d\lambda^d \lesssim \frac{1}{\lambda} \int_{\mathbb{R}^d} |f| \chi_{|f| \geq \lambda/8}\, d\lambda^d.$$

As a consequence, we conclude that

$$\lambda^d(\{|T_\epsilon f| > \lambda\}) \lesssim_{d,T} \frac{1}{\lambda^2} \int_{|f| \leq \lambda} |f|^2\, d\lambda^d + \frac{1}{\lambda} \int_{|f| \geq \lambda/8} |f|\, d\lambda^d,$$

uniformly in $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, $\lambda > 0$, and $0 < \epsilon < 1$.

We now integrate this distributional inequality over the range $\lambda > c_0$, where $0 < c_0 \ll 1$ is some small constant that we will need later:

$$\int_{c_0}^{\infty} \lambda^d(\{|T_\epsilon f| > \lambda\})\, d\lambda \lesssim_{d,T} \int_{\mathbb{R}^d} \int_{c_0}^{\infty} \frac{1}{\lambda^2} \chi_{|f| \le \lambda} |f|^2 \, d\lambda \, d\lambda^d + \int_{\mathbb{R}^d} \int_{c_0}^{\infty} \frac{1}{\lambda} \chi_{\lambda/8 \le |f|} |f| \, d\lambda \, d\lambda^d,$$

and we compute this by simple calculus to be

$$\int_{c_0}^{\infty} \lambda^d(\{|T_\epsilon f| > \lambda\})\, d\lambda \lesssim_{d,T} \int_{\mathbb{R}^d} \frac{|f|^2}{\max(c_0, |f|)} \, d\lambda^d + \int_{|f| \ge c_0/8} |f| \log(8|f|/c_0) \, d\lambda^d$$

so that if $E \subseteq \mathbb{R}^d$ has finite Lebesgue measure, we conclude that

$$\int_E |T_\epsilon f| \, d\lambda^d \lesssim_{d,T} c_0 \lambda^d(E) + (1 + \log(8/c_0)) \int_{\mathbb{R}^d} |f| \, d\lambda^d + \int_{|f| \ge 1} |f| \log |f| \, d\lambda^d,$$

which shows the local integrability of the $T_\epsilon f$ in a manner that depends only on the L^1 and $L \log^+ L$ norms.

We now claim that if $h \in L^1 \cap L \log^+ L$, it is possible to approximate h by elements of $C_c^\infty(\mathbb{R}^d)$ — simultaneously, in both norms (functionals) [2].

To do this, we first take $R \gg 1$ so large that

$$\int_{|x| > R/2} |h(x)| \, dx + \int_{|x| > R/2} |h(x)| \log(1 + |h(x)|) \, dx < \epsilon/4.$$

We then claim it is possible to select a simple function $h_k : B(0, R) \to \mathbb{C}$ to approximate $h\chi_{B(0,R)}$, such that

$$\int_{B(0,R)} |h(x) - h_k(x)|(1 + \log(1 + |h(x) - h_k(x)|)) \, dx < \epsilon/4.$$

Indeed, we take the standard truncated dyadic undergraph approximations h_k of $h\chi_{B(0,R)}$, with $|h_k| \le |h|\chi_{B(0,R)}$ and such that $|h_k - h\chi_{B(0,R)}|$ is decreasing pointwise everywhere to zero. Then, as x and $x \log(1 + x)$ are both monotone increasing on $(0, \infty)$, dominated convergence shows that for sufficiently-large k, relevant integrals will be $< \epsilon/8$ each.

Finally, we convolve, noting that h_k is qualitatively-bounded and nonzero only in a bounded subset, and so in $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. It follows that if we write $x \log(1 + x) \le Cx^2$ on $(0, \infty)$, we can ensure that

$$(C + 1) \int_{\mathbb{R}^d} |h_\delta(x) - h_k(x)|(1 + \log(1 + |h_\delta(x) - h_k(x)|)) \, dx < \epsilon/4,$$

using the x^2 approximation to control the $L\log^+ L$ term by L^2.

We compute easily that

$$\int_{\mathbb{R}^d} |h(x) - h_\delta(x)|\, dx \le \int_{B(0,R)} |h(x) - h_\delta(x)|\, dx + \int_{|x|>R} |h(x)| + |h_\delta(x)|\, dx;$$

the second term contributes $< \epsilon/4$ from the h integral and a negligible $o(1)$ from the h_δ, due to the boundedness of h_k and the algebraic support properties on the convolution, in the process of $\delta \downarrow 0$. Meanwhile, the first term can be divided into $(h - h_k) + (h_k - h_\delta)$, and contributes $< \epsilon/2$.

We estimate the $L\log^+ L$ term similarly; we have a contribution of

$$\le \int_{B(0,R)} |h(x) - h_\epsilon(x)| \log(1 + |h(x) - h_\delta(x)|)\, dx,$$

and here we write

$$\log(1 + |h(x) - h_\delta(x)|) \le \log(1 + |h(x) - h_k(x)|) + \log(1 + |h_k(x) - h_\delta(x)|),$$

by the triangle inequality. The $h - h_k$ and $h_k - h_\delta$ direct contributions are both $< \epsilon/4$, and for $|h_k - h_\delta| \log(1 + |h - h_k|)$, we note the bounded convergence to zero in the first coefficient as $\delta \downarrow 0$, while the second term is integrable on $B(0,R)$. The $|h - h_k| \log(1 + |h_k - h_\delta|)$ term vanishes for the same reason.

We control the outer component by

$$\le \int_{|x|>R} |h(x)| \log(1 + |h(x)|)\, dx + \int_{|x|>R} |h_\delta(x)| \log(1 + |h(x)|)\, dx,$$

and the first term is small by construction, while the second converges boundedly to zero almost-everywhere. The $(|h| + |h_\delta|) \log(1 + |h_\delta|)$ contribution, with the latter term being boundedly-convergent to zero, is even easier.

As a consequence, we have simultaneous control of h in both norms. For our annulus-restricted homogeneous kernel $F\chi_A$, which is in $L^1 \cap L\log^+ L$ (but not necessarily in L^2), we will obtain what we desire via an approximation argument.

Let $(F_n) \subseteq C_c^\infty(2A)$ be a sequence of test functions, converging as $F_n \to F\chi_A$ in the $L^1 \cap L\log^+ L$ norm (its existence follows from what was just shown).

Our inequality for $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ shows that for any ball $B \subseteq \mathbb{R}^d$,

$$\int_B |\vec{T}_\epsilon F_n|\, d\lambda^d \lesssim_{d,B} c_0 + \log(25/c_0) \int_{\mathbb{R}^d} |F_n|\, d\lambda^d + \int_{\mathbb{R}^d} |F_n| \log(1 + |F_n|)\, d\lambda^d,$$

and we notice that our approximation gives that this is uniformly bounded by

$$\lesssim_{d,B} c_0 + \log(25/c_0) \int_{\mathbb{R}^d} |F|\chi_A\, d\lambda^d + \int_{\mathbb{R}^d} |F|\chi_A \log(1 + |F|\chi_A)\, d\lambda^d.$$

We know that as $F_n \to F\chi_A$ in L^1 norm, $\vec{T}_\epsilon F_n \to \vec{T}_\epsilon(F\chi_A)$ in $L^{1,\infty}$ quasinorm, by the weak-type $(1,1)$ inequality on the $\vec{T}_\epsilon$. In particular, we have convergence in measure, and it follows that we can extract a pointwise subsequence which agrees almost everywhere with the operator output in this quasinormed space.

By Fatou's lemma, the limit object $\vec{T}_\epsilon(F\chi_A) \in L^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d)$ actually belongs to $L^1_{\text{loc}}(\mathbb{R}^d;\mathbb{R}^d)$, and obeys the same bounds. Consequently, we can talk about $\vec{T}_\epsilon(F\chi_A)$ (for any $0 < \epsilon < 1$) as a genuine absolutely-integrable function, and handle and compute with it in local L^1 seminorms.

It follows that if $G \in C_c^\infty$ approximates $F\chi_A$ in L^1 and $L\log^+ L$ norm, in the sense that was discussed above, we can consider

$$\int_{r\le|x|\le R} |F_1^\epsilon(x) - F_1^{\epsilon'}(x)|\,dx = \int_{r\le|x|\le R} |(\vec{T}_\epsilon(F\chi_A))(x) - (\vec{T}_{\epsilon'}(F\chi_A))(x)|\,dx$$
$$= \int_{r\le|x|\le R} |(\vec{T}_\epsilon(F\chi_A))(x) - (\vec{T}_\epsilon G)(x)|\,dx + \int_{r\le|x|\le R} |(\vec{T}_\epsilon G)(x) - (\vec{T}_{\epsilon'} G)(x)|\,dx$$
$$+ \int_{r\le|x|\le R} |(\vec{T}_{\epsilon'} G)(x) - (\vec{T}_{\epsilon'}(F\chi_A))(x)|\,dx,$$

and we get that

$$\lim_{\epsilon,\epsilon'\downarrow 0} \int_{r\le|x|\le R} |(\vec{T}_\epsilon G)(x) - (\vec{T}_{\epsilon'} G)(x)|\,dx = 0,$$

due to the singular integrals converging in $L^p(\mathbb{R}^d)$ norm for all $1 < p < \infty$, and hence in $L^1_{\text{loc}}(\mathbb{R}^d)$, by the maximal-operator theory.

The remaining two integrals are controlled uniformly in $0 < \epsilon, \epsilon' < 1$ by

$$\lesssim_{d,r,R} c_0 + \log(25/c_0) \int_{\mathbb{R}^d} |F\chi_A - G|\,d\lambda^d + \int_{\mathbb{R}^d} |F\chi_A - G| \log(1 + |F\chi_A - G|)\,d\lambda^d,$$

by our above; optimizing over $G \in C_c^\infty(\mathbb{R}^d)$, we obtain a final $\lesssim_{d,r,R} c_0$, where $0 < c_0 \ll 1$ was arbitrary. We conclude that $\limsup_{\epsilon,\epsilon'\downarrow 0} \|F_1^\epsilon - F_1^{\epsilon'}\|_{L^1(B_R\setminus B_r)} = 0$ for all $r, R > 0$, showing the desired convergence in $L^1_{\text{loc}}(\mathbb{R}^d \setminus \{0\})$.

To obtain the convergence of F_2^ϵ, we do the same thing and write

$$F_2^\epsilon(x) - F_2^{\epsilon'}(x) = (\vec{T}_\epsilon(F\phi))(x) - (\vec{T}_{\epsilon'}(F\phi))(x) = (\vec{T}_\epsilon(F\varphi))(x) - (\vec{T}_{\epsilon'}(F\varphi))(x)$$

for $|x| \le R$ (we need not avoid $x = 0$), and the result follows once more.

Now, to control the difference between the two functions, we write

$$F_1^\epsilon(x) - F_2^\epsilon(x) = \int_{|x-y|>\epsilon} \chi_{|y|\ge\delta_x} \vec{K}(x-y)F(y)(1-\phi(y))\,dy$$
$$+ \lim_{\delta\downarrow 0} \int_{|x-y|>\epsilon} \chi_{\delta<|y|<\delta_x} \vec{K}(x-y)F(y)(1-\phi(y))\,dy,$$

and we note that for small $\epsilon > 0$ (relative to x), the second term can be written

$$\lim_{\delta \downarrow 0} \int_{\mathbb{R}^d} \chi_{\delta < |y| < \delta_x} \vec{K}(x-y) F(y)(1-\phi(y))\, dy$$
$$= \int_{\mathbb{R}^d} \chi_{|y| < \delta_x} (\vec{K}(x-y) - \vec{K}(x)) F(y)(1-\phi(y))\, dy,$$

while we express the first term as

$$\int_{|x-y|>\epsilon} \chi_{|y| \geq \delta_x} F(y)(1-\phi(y)) \vec{K}(x-y)\, dy$$
$$= \int_{\mathbb{R}^d} \chi_{\delta_x \leq |y| \leq \frac{3}{4}} F(y)(1-\phi(y)) \vec{K}(x-y)\, dy$$
$$= \int_{\mathbb{R}^d} \chi_{\delta_x \leq |y| \leq \frac{3}{4}} F(y)(1-\phi(y))(\vec{K}(x-y) - \vec{K}(x))\, dy$$

in the $\epsilon \downarrow 0$ limit, using that $|x| \geq 1$ while $|y| \leq \frac{3}{4}$ in here, so the $y = x$ singularity is removed.

As a consequence,

$$|F_1(x) - F_2(x)| \leq \int_{|y| \leq \frac{3}{4}} |F(y)||\vec{K}(x-y) - \vec{K}(x)|\, dy,$$

and we use the smoothness of $\vec{K}$ to get a bound of $\lesssim_d \|F\|_{L^1(S^{d-1})} |x|^{-d-1}$. This shows the desired estimate for $|x| \geq 1$.

Now, we proceed to dominate and estimate F_2. First, if $|x| < \frac{1}{8}$, we can write

$$F_2(x) = \lim_{\epsilon \downarrow 0} \int_{|x-y|>\epsilon} |\vec{K}(x-y)||F(y)|\phi(y) \lesssim_d \int_{\mathbb{R}^d} |y|^{-d} |F(y)|\phi(y)\, dy,$$

using that $F\phi$ vanishes in $B(0, 1/4)$. Taking polar coordinates, this can be estimated by $\|F\|_{L^1(S^{d-1})} \int_{1/4}^{\infty} r^{-2d} r^{d-1}\, dr$, and we conclude that for small x

$$|F_2(x)| \leq C_d \|F\|_{L^1(S^{d-1})}.$$

For $\frac{1}{8} < |x| < 1$, we have

$$|F_2(x)| \leq \phi(x)|F_1(x)| + |F_2(x) - \phi(x) F_1(x)|,$$

and the latter term can be written as

$$\lim_{\epsilon \downarrow 0} \lim_{\delta \downarrow 0} \int_{|x-y|>\epsilon} \vec{K}(x-y) F(y)\phi(y) - \chi_{|y|>\delta} \vec{K}(x-y) F(y) \phi(x)\, dy,$$

or

$$\lim_{\epsilon \downarrow 0} \lim_{\delta \downarrow 0} \int_{|x-y|>\epsilon} \chi_{|y|>\delta} \vec{K}(x-y) F(y)(\phi(y) - \phi(x))\, dy,$$

and we use the mean-zero condition to write this as

$$\int_{|x-y|>\epsilon} \chi_{|y|>1} \vec{K}(x-y)F(y)(\phi(y)-\phi(x))\,dy$$
$$+\int_{|x-y|>\epsilon} \chi_{|y|\le 1}(\vec{K}(x-y)-\vec{K}(x))F(y)(\phi(y)-\phi(x))\,dy,$$

taking the $\delta \downarrow 0$ limit.

Now, we can apply the triangle inequality to write this as

$$\le \int_{\mathbb{R}^d} |\vec{K}(x-y)-\chi_{|y|<1}\vec{K}(x)||\phi(y)-\phi(x)||F(y)|\,dy.$$

We claim that for all $\frac{1}{8}<|x|<1$, and $y\in\mathbb{R}^d$, we have

$$|\vec{K}(x-y)-\chi_{|y|<1}\vec{K}(x)| \lesssim |y|^{1/2}|x-y|^{-d}.$$

Indeed, if $|y|\le \frac{1}{16}<\frac{1}{2}|x|$, we would have

$$|\vec{K}(x-y)-\vec{K}(x)| \lesssim |y|/|x|^{d+1} \lesssim |y| \le |y|^{1/2} \lesssim |y|^{1/2}/|x-y|^d.$$

For $\frac{1}{16}<|y|<1$, we would have

$$|\vec{K}(x-y)-\vec{K}(x)| \lesssim |x-y|^{-d}+|x|^{-d} \lesssim |x-y|^{-d}+1 \lesssim |y|^{1/2}(|x-y|^{-d}+1)$$

and since $|x-y|\le|x|+|y|\le 1$, we have $1\lesssim |x-y|^{-d}$. As a consequence, we obtain the desired estimate.

Finally, if $|y|\ge 1$, we simply have $|\vec{K}(x-y)|\lesssim |x-y|^{-d}\le |y|^{1/2}|x-y|^{-d}$.

We consequently bound this by

$$|F_2(x)| \lesssim |F_1(x)| + \int_{\mathbb{R}^d} |y|^{1/2}|F(y)||x-y|^{-(d-1)}\,dy.$$

Since $\frac{1}{8}\le|x|\le 1$, this can be further controlled by

$$|F_2(x)| \lesssim \|F\|_{L^1(S^{d-1})} + |x|^d|F_1(x)| + |x|^{d-3/2}\int_{\mathbb{R}^d} |y|^{1/2}|F(y)||x-y|^{-(d-1)}\,dy,$$

and we can recognize that the right-hand side is homogeneous of degree 0.

To estimate the L^1 norm when restricted to S^{d-1}, we simply take the integral in x for $\frac{1}{2}\le|x|\le 1$, say; the first term is constant, and the second produces a contribution of $\int_{|x|\sim 1}|F_1(x)|\,dx$; we saw that $F_1\in L^1_{\text{loc}}(\mathbb{R}^d\setminus\{0\})$, and the constant will be some term depending on F.

For the final term, we note that before linearly integrating in $|x|\sim\frac{1}{2}$, we get

$$\lesssim \int_{|y|\le 4} |y|^{1/2}|F(y)||x-y|^{-(d-1)}\,dy + \int_{|y|>4} |y|^{1/2}|F(y)||y|^{-(d-1)}\,dy;$$

the singularity $|\cdot|^{-(d-1)}$ is locally-integrable, so integrating over x in the first term, we have a bound of

$$\lesssim_d \int_0^4 \int_{S^{d-1}} r^{1/2} r^{-d} |F(\omega)| r^{d-1}\, d\sigma\, dr \lesssim \|F\|_{L^1(S^{d-1})}.$$

For the far-range term, we see that it is also $\lesssim \|F\|_{L^1(S^{d-1})}$.

As a consequence, we get that

$$\|G\|_{L^1(S^{d-1})} \lesssim_{d,F} 1,$$

as desired.

Finally, we note that if F is better than $L\log^+ L$ on S^{d-1}, we can mostly avoid this nonsense by using the simpler theory of the $L^q(\mathbb{R}^d)$ convergence of truncated singular integrals, $q > 1$; we have maximal boundedness of the $\vec{T}_\epsilon(F\chi_A)$, since $F\chi_A \in L^q$; we then get $F_1 \in L^q_{\text{loc}}(\mathbb{R}^d \setminus \{0\})$, and $F_2 \in L^q_{\text{loc}}(\mathbb{R}^d)$, and convergence in those local seminorms as $\epsilon \downarrow 0$.

We will have $F\phi \in L^q(\mathbb{R}^d)$, $F\chi_{B^c} \in L^q$ whenever B encloses the origin, so we can estimate F_1 by taking

$$|F_1(x)| \leq (\vec{R}_*(F\chi_{B^c}))(x) + \int_{|y|<\delta_x} |F(y)||\vec{K}(x-y) - \vec{K}(x)|\, dy,$$

where $B = B(0, \delta_x) = B(0, \frac{r}{8}) = B(0, \frac{1}{256})$, say (where $|x| \sim 1$).

Using smoothness, the second term with $|x| \sim 1$ is uniformly bounded by $\|F\|_{L^1(S^{d-1})}$, so it only remains to take qth powers of the first term and investigate the effect; we can use the standard Cotlar's inequality (or even Theorem III, applied to $\vec{K}$), to obtain a bound of $\lesssim_{d,q} \|F\chi_{B^c}\|_{L^q(\mathbb{R}^d)} \simeq_{d,q} \|F\|_{L^q(S^{d-1})}$.

By Hölder's inequality, we may control the first contribution by $L^q(S^{d-1})$, and so we have the upper-bound on $\|F_1\|_{L^q(S^{d-1})}$. Then, using this very inequality, the L^1 estimate of G goes through, but with $\|F\|_{L^q(S^{d-1})}$ concretely on the right-hand side (again applying Hölder).

This all means that, with the estimate for $\|F_1\|_{L^q(S^{d-1})}$ in hand, we now have control of the first two terms of G in the integral $\|G\|_{L^q(S^{d-1})}$. For the final term, we argue as follows: assuming $\frac{1}{2} \leq |x| \leq 1$, we have

$$|x|^{d-3/2} \int_{|y|<\frac{1}{4}} |y|^{1/2} |F(y)||x-y|^{-(d-1)}\, dy$$
$$+ |x|^{d-3/2} \int_{\frac{1}{4} \leq |y| \leq 2} |y|^{1/2} |F(y)||x-y|^{-(d-1)}\, dy$$
$$+ |x|^{d-3/2} \int_{|y|>2} |y|^{1/2} |F(y)||x-y|^{-(d-1)}\, dy,$$

and the first two terms are seen to contribute

$$\lesssim |x|^{d-3/2}\left(\int_{|y|<\frac{1}{4}} |y|^{1/2}|F(y)||x|^{-(d-1)}\,dy + \int_{|y|>2} |y|^{1/2}|F(y)||y|^{-(d-1)}\,dy\right).$$

For the second term, since $d > 1$ (there is no such thing as an even function with mean-zero on the sphere in $\mathbb{R}^1$, other than 0), we use the Hardy-Littlewood-Sobolev inequality; the term is a fractional integral operator, the $\alpha = 1$ Riesz potential I_1, of the function $J = |\cdot|^{1/2}\chi_{\frac{1}{4}\leq|\cdot|\leq 2}|F| \in L^q(\mathbb{R}^d)$.

First consider if $q > 1$ in the definition satisfies $q < d$. Then we can straightforwardly apply the HLS inequality, and obtain that $I_1(J) \in L^r(\mathbb{R}^d)$ for the index defined by $\frac{1}{r} = \frac{1}{q} - \frac{1}{d}$. We may then take the $\|\cdot\|_{L^q(B(0,1))}$ norm in x, then use Hölder's inequality to increase to the L^r norm before using HLS to obtain the L^q norm of J, which is $\simeq \|F\|_{L^q(S^{d-1})}$.

For $q \geq d$, we know (by the compact support of J) that $J \in L^{\tilde{q}}(\mathbb{R}^d)$ for $\tilde{q} = \frac{d}{1+\epsilon}$, for any $0 < \epsilon < d-1$. As a consequence, the output of the Riesz potential is in $L^{d/\epsilon}(\mathbb{R}^d)$ for all small $\epsilon > 0$, and it follows that we can have it meet (or exceed) the value of $d \leq q < \infty$; taking the L^q norm in x, we exchange for the $L^{\frac{d}{\epsilon}}$ norm, and then return to the $L^{\tilde{q}}$ norm, where we know $\tilde{q} \leq q$.

Consequently, as a result of this averaging over $|x| \sim 1$, we can say that

$$\|G\|_{L^q(S^{d-1})} \lesssim_{d,q} \|F\|_{L^q(S^{d-1})},$$

and this completes the final piece of the estimate for the F-regular case. $\square$

We now proceed to begin the proof of Theorem I. We take $N : \mathbb{R}^d \setminus \{0\} \to \mathbb{C}$ satisfying the stated hypotheses, $N(-x) = N(x)$ (even, as we may assume, by our reductions), and take

$$N_1(x) = \lim_{\epsilon\downarrow 0}\lim_{\delta\downarrow 0}\int_{|x-y|>\epsilon} \chi_{|y|>\delta}\vec{K}(x-y)N(y)\,dy$$

as before. We know that the convergence will be in L^1_{loc} on $\mathbb{R}^d \setminus \{0\}$, and now N_1 is odd: $N_1(-x) = -N_1(x)$.

Likewise, we get

$$N_2(x) = \lim_{\epsilon\downarrow 0}\int_{|x-y|>\epsilon} \vec{K}(x-y)N(y)\phi(y)\,dy,$$

and by our Claims, we will have $|N_1(x) - N_2(x)| \lesssim |x|^{-d-1}$ for $|x| \geq 1$, and $|N_2(x)| \leq G(x)$ when $|x| \leq 1$, for $G \in L^1(S^{d-1})$ and degree 0.

Now, consider a function $g : \mathbb{R}^d \to \mathbb{C}^d$, with $|g| \in L^p(\mathbb{R}^d)$. We then know that the modified scalar Riesz transform

$$f(x) = \lim_{\epsilon\downarrow 0}\int_{|x-y|>\epsilon} \langle\vec{K}(x-y), \vec{g}(y)\rangle\,dy$$

exists pointwise almost everywhere, and converges in L^p norm also. Moreover, by our earlier discussion, every function $f \in L^p(\mathbb{R}^d)$ arises in this manner.

We now claim the following: if $f \in L^p(\mathbb{R}^d)$ is of the form described, then we have the representation

$$\int_{\mathbb{R}^d} N(x-y)\phi((x-y)/\epsilon)f(y)\,dy = \epsilon^{-d}\int_{\mathbb{R}^d} \langle N_2((x-y)/\epsilon), \vec{g}(y)\rangle\,dy,$$

recalling that N_1 and N_2, as vector Riesz transforms of scalar functions, are themselves vector-valued.

We note that because $N\phi_\epsilon$ is a homogeneous $L^1(S^{d-1})$ kernel, and f lies in $L^p(\mathbb{R}^d)$, the calculations we did in Section II show that the left-hand side is finite almost everywhere. For the right, we can directly use the bounds on N_2 near the origin and N_1 and $N_1 - N_2$ away from it, along with Section II once more, to conclude that the right-hand side is finite also.

We first consider vector-valued $\vec{g} \in C_c^\infty(\mathbb{R}^d;\mathbb{R}^d)$, and note that

$$\int_{\mathbb{R}^d} N(x-y)\phi((x-y)/\epsilon)\left(\int_{|y-z|>\delta}\langle \vec{K}(y-z), \vec{g}(z)\rangle\,dz\right)dy,$$

and we note that upon taking absolute values, switching to polar coordinates in y and using the compact support and boundedness of $\vec{g}$ to obtain additional decay for large r from the off-support singular integral, integrating in $x \in B$, we see that this integral is absolutely convergent for a.e. x.

So for a.e. x, we may apply Fubini, and write

$$\int_{\mathbb{R}^d}\left[\int_{|y-z|>\delta} N(x-y)\phi((x-y)/\epsilon)\vec{K}(y-z)\,dy\right]\cdot \vec{g}(z)\,dz.$$

Then we can change variables to see that this is

$$\int_{|x-y-z|>\delta} N(y)\phi(y/\epsilon)\vec{K}(x-y-z)\,dy,$$

and by homogeneity of all the kernels, this is equal to

$$\epsilon^d\int_{|(x-z)/\epsilon-y|>\delta/\epsilon} \epsilon^{-d}N(y)\phi(y)\vec{K}((x-z)/\epsilon-y)\epsilon^{-d}\,dy,$$

or

$$\epsilon^{-d}N_2^{\delta/\epsilon}((x-z)/\epsilon).$$

We know generally that $N_2^\epsilon \to N_2$ as $\epsilon \downarrow 0$, by the Claim above. Moreover, we saw that we have L^1_{loc} convergence to the limit. Since $\vec{g} \in C_c^\infty$, we conclude that as $\delta \downarrow 0$, the whole integral converges to

$$\int_{\mathbb{R}^d}\epsilon^{-d}\langle N_2((y-z)/\epsilon), \vec{g}(z)\rangle\,dz,$$

for pointwise a.e. x.

Meanwhile, the integral defining f can be represented by

$$\int_{|y-z|>\delta} \langle \vec{K}(y-z), \vec{g}(z) \rangle \, dz = \int_{|y-z|>\delta} \chi_{|y-z|<1} \langle \vec{K}(y-z), \vec{g}(z) - \vec{g}(y) \rangle \, dz + \int_{|y-z|\geq 1} \langle \vec{K}(y-z), \vec{g}(z) \rangle \, dz,$$

using the mean-zero property of the Riesz kernel. As a consequence, this representation (which we will only use for this purpose) shows that the truncated integrals for $\delta \downarrow 0$ converge pointwise everywhere to a limit.

Next, we see the difference between the truncated integral and f is given by

$$\int_{|y-z|<\delta} \langle \vec{K}(y-z), \vec{g}(z) \rangle \, dz,$$

which, by pointwise convergence, is majorized by $\leq 2\tilde{R}_*(\vec{g}) \in L^p(\mathbb{R}^d)$.

Again recalling the discussion in Section 2, we see that

$$\begin{aligned} &\left\| \int_{\mathbb{R}^d} N(x-y)\phi((x-y)/\epsilon)(f - f_\delta)(y) \, dy \right\|_{L^1_x(B)} \\ &\leq \left\| \int_{|x-y|\gtrsim \epsilon} |N(x-y)||f(y) - f_\delta(y)| \, dy \right\|_{L^1_x(B)} \\ &\lesssim_{d,p,\epsilon,B,N} \|f - f_\delta\|_{L^p(\mathbb{R}^d)} \to 0 \end{aligned}$$

as $\delta \downarrow 0$, by dominated convergence and the L^p boundedness of the singular maximal operator. It follows that for a.e. x, after passing to a subsequence, we have convergence to zero; and so the desired identity holds pointwise almost everywhere when $\vec{g} \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^d)$.

To pass to general $\vec{g} \in L^p(\mathbb{R}^d; \mathbb{R}^d)$, we let $(\vec{g_n}) \subseteq C_c^\infty(\mathbb{R}^d; \mathbb{R}^d)$ be a sequence converging to $\vec{g}$ in $L^p(\mathbb{R}^d; \mathbb{R}^d)$ norm.

We can see that

$$\int_{\mathbb{R}^d} N(x-y)\phi((x-y)/\epsilon) f_n(y) \, dy = \int_{\mathbb{R}^d} \epsilon^{-d} \langle N_2((x-z)/\epsilon), \vec{g_n}(z) \rangle \, dz,$$

and we would like to take limits $n \to \infty$ on both sides of this. If we replace $\vec{g_n}$ by $\vec{g}$ and f_n by f, both sides will be in $L^1_{\text{loc}}(\mathbb{R}^d)$, by our arguments in Section 2 (the integrals are absolutely convergent almost-everywhere), but we do not know if the two functions they define are equal.

Yet we know that $\vec{g_n} - \vec{g} \to 0$ in $L^p(\mathbb{R}^d; \mathbb{R}^d)$, so it follows (by using the same estimates we employed in Section 2, to show finiteness) that we can bound

$$\|\text{RHS}(\vec{g_n}) - \text{RHS}(\vec{g})\|_{L^1(B)} \lesssim_{d,p,B,N} \|\vec{g_n} - \vec{g}\|_{L^p(\mathbb{R}^d;\mathbb{R}^d)} \to 0.$$

By passing to a subsequence, there exists a full-measure set of x for which the convergence on the left-hand side holds pointwise. In the same vein, we have

$$\|\mathrm{LHS}(f_{n_k}) - \mathrm{LHS}(f)\|_{L^1(B)} \lesssim_{d,p,B,N} \|f_n - f\|_{L^p(\mathbb{R}^d)} \lesssim_{d,p} \|\vec{g_n} - \vec{g}\|_{L^p(\mathbb{R}^d;\mathbb{R}^d)},$$

and we can also pass to a further subsequence here. Finally, once we exclude the countably-many null sets where $(\mathrm{LHS}(f_n))(x) \neq (\mathrm{RHS}(g_n))(x)$, we obtain that the identity holds a.e., for general $\vec{g} \in L^p$.

We now proceed by observing that

$$\begin{aligned}(T_\epsilon f)(x) = \int_{\mathbb{R}^d} & N(x-y)\phi((x-y)/\epsilon)f(y)\,dy \\ &+ \int_{\mathbb{R}^d} N(x-y)(\chi_{|x-y|>\epsilon} - \phi((x-y)/\epsilon))f(y)\,dy,\end{aligned}$$

and the latter integral can be represented as

$$\int_{\mathbb{R}^d} \epsilon^{-d} N((x-y)/\epsilon)\psi((x-y)/\epsilon)f(y)\,dy,$$

where we can see that $\psi \in C_c^\infty(\mathbb{R}^d;[0,1])$ is a radial cutoff localizing to the annulus $\mathrm{supp}(\psi) \subseteq \{\frac{1}{4} \leq |\cdot| \leq 1\}$.

Now, we can take absolute values, and write this as

$$\begin{aligned} &\lesssim \epsilon^{-d} \int_{\epsilon/4 \leq |x-y| \leq \epsilon} |N((x-y)/\epsilon)||f(y)|\,dy \\ &= \epsilon^{-d} \int_{\epsilon/4}^{\epsilon} \int_{S^{d-1}} |N(r\omega/\epsilon)||f(x-r\omega)|\,d\sigma\, r^{d-1}\,dr \\ &= \epsilon^{-d} \int_{\epsilon/4}^{\epsilon} \int_{S^{d-1}} (\epsilon/r)^d |N(\omega)||f(x-r\omega)|\,d\sigma\, r^{d-1}\,dr \\ &\leq 4^d \epsilon^{-d} \int_{\mathbb{R}^d} N((x-y)/|x-y|)\chi_{|\,\cdot\,|>1}((x-y)/\epsilon)|f(y)|\,dy, \end{aligned}$$

and recognizing the degree 0 term, Theorem V then says this is bounded uniformly in $\epsilon > 0$ by a maximal function which maps into $L^p(\mathbb{R}^d)$.

The main term can be bounded in absolute value by

$$\begin{aligned} &\left| \int_{\mathbb{R}^d} \epsilon^{-d} \langle N_2((x-z)/\epsilon), (\vec{R}f)(z)\rangle\,dz \right| \\ \lesssim &\left| \int_{|x-z|>\epsilon} \epsilon^{-d} \langle N_1((x-z)/\epsilon), (\vec{R}f)(z)\rangle\,dz \right| \\ &+ \epsilon^{-d} \int_{|x-z|>\epsilon} |(x-y)/\epsilon|^{-d-1} |(\vec{R}f)(z)|\,dz \\ &+ \epsilon^{-d} \int_{|x-z|\leq\epsilon} G((x-z)/\epsilon)|(\vec{R}f)(z)|\,dz. \end{aligned}$$

Now, the final term is also treatable via Theorem V, G being mean-zero, and it gives control by a standard Hardy-Littlewood type maximal function.

The second term is L^p bounded by the radial majorant lemma, thanks to the good integrability properties of the kernel $|\cdot|^{-d-1}\chi_{|\cdot|>1}$.

By homogeneity, the first term is simply equal to convolution with $N_1\chi_{|\cdot|>\epsilon}$, which produces a truncated singular integral precisely of the form that can be handled by Theorem III; F_1 is odd.

From the calculations in Section 2, we see that there exists a set of full measure $N^c \subseteq \mathbb{R}^d$ such that for all $x \in N^c$, $\int_{|x-y|>\epsilon} |K(x,y)||f(y)|\,dy < \infty$ for all $\epsilon > 0$.

Indeed, we saw this occurred whenever $\int_{S^{d-1}} |N(\omega)|(\int_{\mathbb{R}} |f(x+r\omega)|^p\,dr)^{1/p}\,d\sigma < \infty$, and by integration, we saw that this quantity was finite for a.e. $x \in \mathbb{R}^d$.

Dominated convergence shows $\epsilon \mapsto |\int_{|x-y|>\epsilon} K(x,y)f(y)\,dy|$ is continuous in $\epsilon \in (0,\infty)$ at these points, so it follows that we can restrict the supremum in $\epsilon > 0$, off a null set of x, to the dyadic rationals.

From this, we have Borel measurability of the full maximal function. Moreover, we see the right-hand side bounding T_*f comprises three maximal functions, and each in $L^p(\mathbb{R}^d)$, by our previous Theorems.

It follows that we can take the supremum and obtain the maximal estimate

$$\|T_*f\|_{L^p(\mathbb{R}^d)} \lesssim_{d,p} \|f\|_{L^p(\mathbb{R}^d)}.$$

Finally, we consider the $\epsilon \downarrow 0$ limit for $T_\epsilon f$, when $f \in C_c^\infty(\mathbb{R}^d)$. We write

$$(T_\epsilon f)(x) = \int_{|x-y|\geq 1} N(x-y)f(y)\,dy + \int_{\epsilon<|x-y|<1} N(x-y)(f(y)-f(x))\,dy,$$

using the mean-zero property of N on every sphere centered at the origin. It follows that the second term is an absolutely convergent integral, and we may take $\epsilon \downarrow 0$, to thus obtain a measurable function Tf.

By Fatou's lemma, we have $Tf \in L^p(\mathbb{R}^d)$, and by the L^p boundedness of the maximal function, we also have convergence in L^p. The standard argument also shows that $T_\epsilon f \to Tf$ for all $f \in L^p(\mathbb{R}^d)$, pointwise a.e.

This proves Theorem I.

For Theorem II, we note the following: we reduced to the case where $N(x,\cdot)$ is uniformly in $L^q(S^{d-1})$ for $1 < q < \infty$, and we produce the vector-valued kernel functions (defined for $y \neq 0$)

$$N_1(x,y) = \lim_{\epsilon\downarrow 0}\lim_{\delta\downarrow 0}\int_{|y-z|>\epsilon} N(x,z)\chi_{|z|>\delta}\vec{K}(y-z)\,dz,$$

$$N_2(x,y) = \lim_{\epsilon\downarrow 0}\int_{|y-z|>\epsilon} N(x,z)\phi(z)\vec{K}(y-z)\,dz.$$

We apply our previous Claim to obtain these kernels, treating $N(x,z) = N_x(z)$ as an x-indexed family of functions of z. To see product-measurability, we see that each integral in z, for fixed $\epsilon > 0$ and $\delta > 0$, is product-measurable, as a slice integral by Fubini, with the finiteness coming again from the polar coordinates calculation.

In taking the $\delta \downarrow 0$ limit, we saw that the limit existed for all $y \neq 0$ and $x \in \mathbb{R}^d$; thus the fixed-ϵ integral is a product measurable function in (x,y), as the pointwise limit of a sequence of such functions.

Then, to take the $\epsilon \downarrow 0$ limit, we not only integrate in $r < |y| < R$ but also $x \in B$; it gives that the difference in $L^q_{\mathrm{loc}}(\mathbb{R}^d \setminus \{0\})$ equals

$$\int_B \left(\int_{r<|y|<R} |(T_\epsilon(N_x\chi_A))(y) - (T_{\epsilon'}(N_x\chi_A))(y)|^q \, dy \right) dx,$$

when integrated.

We see that for each fixed x, $\int F_{\epsilon,\epsilon'}(x,y)\,dy \to 0$ as $\epsilon, \epsilon' \to 0$. Moreover, we have the uniform-in-ϵ pointwise bounds in x of $\int F_{\epsilon,\epsilon'}(x,y)\,dy \lesssim \int |(N_x\chi_A)(y)|^q\,dy \leq C_N$, and we use this last term (as a constant, which is certainly a measurable function of x) to apply dominated convergence.

It follows that this iterated integral tends to 0 in the $\epsilon, \epsilon' \downarrow 0$ limit. But because we saw the integrand is simply the difference of measurable functions $N_1^\epsilon(x,y)$, the iterated integral must equal the product integral, and thus we have

$$\lim_{\epsilon,\epsilon' \downarrow 0} \iint_{B\times\{r<|\cdot|<R\}} |N_1^\epsilon(x,y) - N_1^{\epsilon'}(x,y)|^q \, dx\, dy = 0.$$

From this, we get joint convergence in (x,y), and so we may take a pointwise subsequence to ensure that $N_1^\epsilon(x,y) \to N_1(x,y)$, a product-measurable function in $L^1_{\mathrm{loc}}(\mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}))$, and which is (after suitable corrections) odd and homogeneous in the second variable.

(For a clean approach, which avoids delicate tinkering with pointwise representatives, we can pass to a diagonal subsequence of ϵ and take the Borel set of (x,y) where $\limsup_{N\to\infty} \sup_{n,m\geq N} |N_1^{\epsilon_n}(x,y) - N_1^{\epsilon_m}(x,y)| = 0$; this is the *maximal* set of pointwise convergence, to which we restrict the limit object N_1. This set has full measure, by our diagonalization procedure (comparing with the L^1 object). It also follows that if $(x,y) \in F$, then $(x,\lambda y) \in F$, and also $(x,-y) \in F$; by the true homogeneity and oddness of the approximate kernels, and these identities hold. So, for almost all choices of $x = x_0$, we will have that $y \mapsto N_1(x_0,y)$ is defined for a.e. y, and moreover, this is truly odd and homogeneous: if y is such that $(x_0,y) \in F$, we have all the desired scaling properties; if $(x_0,y) \notin F$, then likewise, so $N_1(x_0,y) = 0 = N_1(x_0,-y) = N_1(x_0,\lambda y)$, recalling our multiplication by the indicator function.)

By the uniformity in x, we immediately obtain the estimate

$$|N_1(x,y) - N_2(x,y)| \lesssim_N |y|^{-d-1}$$

for $|y| \geq 1$, directly from the Claim.

We likewise have a product-measurable function

$$G(x,y) = \|N(x,\cdot)\|_{L^q(S^{d-1})} + |y|^d |N_1(x,y)| \\ + |y|^{d-3/2} \int_{\mathbb{R}^d} |y|^{1/2} |N(x,y)| |x-y|^{-(d-1)} \, dy$$

to majorize N_2 in $|y| \leq 1$, and this is still homogeneous of degree zero, and also in $L^q(S^{d-1})$ in the second variable, uniformly in x.

We repeat the same manipulations as before, this time employing the identity

$$\int_{\mathbb{R}^d} N(x, x-y) \phi((x-y)/\epsilon) f(y) \, dy = \epsilon^{-d} \int_{\mathbb{R}^d} \langle N_2(x, (x-y)/\epsilon), \vec{g}(y) \rangle \, dy$$

for $\vec{g} \in L^p(\mathbb{R}^d; \mathbb{R}^d)$, over a.e. x. This proceeds *mutatis mutandis* as Section 2, but this time we apply the uniform-in-x L^q property. Because of these uniform bounds, we eventually see that we do not need to worry about many null sets of x; we have a single null set such that equality holds everywhere outside of it.

Finally, applying the same bounds as before, and using Theorem IV with variable kernel, Theorem VI for the case of variable kernel, we again obtain a uniform estimate of the maximal singular integral by a sum: the maximal operator associated with N_1 (IV), a convolution with $\chi_{|y|>1} |y|^{-d-1} \gtrsim |N_1(x,y) - N_2(x,y)|$, and convolution with $\chi_{|y|\leq 1} G(x,y) \gtrsim |N_2(x,y)|$ (VI).

This completes the proof of Theorem II.

Corollary: Let $\Omega \subseteq \mathbb{R}^d$ be a bounded convex domain. Then there is a bounded operator for $1 < p < \infty$, $T : (L^p(\Omega))_0 \to [W_0^{1,p}(\Omega)]^d$, such that if $\vec{u} = T(f)$,

$$\mathrm{div}(\vec{u}) = f.$$

Proof. Bogovskii's construction [3] furnishes an explicit operator acting on the mean-zero functions in $L^1(\Omega)$, which maps into $L^1(\Omega)$ and recovers the divergence in the distributional sense. Moreover, on smooth mean-zero functions of compact support, this gives a smooth vector field of compact support.

It follows that on smooth functions, the divergence equation is satisfied in the classical sense. Furthermore, it is then the case that the partial derivatives of each component can be expressed as a sum of singular and weakly-singular integral operators, acting on the data f.

Using Theorem II in the case of the relevant second-generation kernel (where it holds for all $1 < q < \infty$), one obtains L^p bounds on the $\partial_i u_j$. $\square$

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