Cancellation for Calderón-Zygmund Kernels

Patrick Li

September 2025

Definition: Let $K : (\mathbb{R}^d \times \mathbb{R}^d)^* \to \mathbb{C}$ be any function satisfying the following conditions: for some constant $C_K \geq 0$ and $0 < \delta \leq 1$,

- For all $x \in \mathbb{R}^d$, $y \in \mathbb{R}^d$,

  $$|K(x, y)| \leq C_K |x - y|^{-d}$$

  (the size condition).

- For all $x, x' \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$ with $|x - x'| \leq \frac{1}{2}|x - y|$, we have

  $$|K(x, y) - K(x', y)| \leq C_K |x - x'|^{\delta}/|x - y|^{d+\delta}$$

  (the smoothness condition).

- Additionally, suppose that when $x \in \mathbb{R}^d$ and $y, y' \in \mathbb{R}^d$ are such that $|y - y'| \leq \frac{1}{2}|x - y|$, then

  $$|K(x, y) - K(x, y')| \leq C_K |y - y'|^{\delta}/|x - y|^{d+\delta}$$

  (the adjoint smoothness condition).

We say that any function satisfying these conditions is a *standard kernel*.

Theorem: Let K be a standard kernel. Suppose that for all $\epsilon > 0$ and $N > 0$, $\epsilon < N$, that the functions

$$I_{\epsilon,N}(x) = \int_{\epsilon < |x-y| < N} K(x, y)\, dy, \qquad I^*_{\epsilon,N}(y) = \int_{\epsilon < |y-x| < N} K(x, y)\, dx$$

satisfy the boundedness conditions

$$\sup_{x_0 \in \mathbb{R}^d} \int_{B(x_0,N)} |I_{\epsilon,N}(x)|\, dx \leq CN^d, \qquad \sup_{y_0 \in \mathbb{R}^d} \int_{B(y_0,N)} |I^*_{\epsilon,N}(y)|\, dy \leq CN^d$$

where the constant C is uniform over all choices of ϵ, N. Then there is a subsequence of the operators $T_{\epsilon,R} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$, defined by

$$(T_{\epsilon,R}(f))(x) = \int_{\epsilon < |x-y| < R} K(x, y) f(y)\, dy,$$

converging weakly to a Calderón-Zygmund operator T with kernel K.

Proof. We will use Tao's formulation of the $T(1)$ theorem. We will verify uniform L^2-boundedness of the truncated operators $T_{\epsilon,R}$.

We note that the condition

$$\int_{B(x_0,N)} |I_{\epsilon,N}(x)|\,dx \lesssim N^d$$

really implies that

$$\int_{B(x_0,N)} |(T_{\epsilon,N}(\chi_{B(x_0,2N)}))(x)|\,dx \lesssim N^d;$$

indeed, for any $x \in B(x_0, N)$,

$$I_{\epsilon,N}(x) = \int_{\epsilon<|x-y|<N} K(x,y)\,dy = \int_{\epsilon<|x-y|<N} K(x,y)\chi_{B(x_0,2N)}(y)\,dy,$$

since $|x-y| < N$ and $|x-x_0| < N$ implies $|y-x_0| < 2N$.

We then write

$$(T_\epsilon f)(x) = \int_{|x-y|>\epsilon} K(x,y)f(y)\,dy,$$

and so we note that

$$(T_{\epsilon,N}(\chi_{B(x_0,2N)}))(x) = (T_\epsilon(\chi_{B(x_0,2N)}))(x) - \int_{|x-y|>N} K(x,y)\chi_{B(x_0,2N)}(y)\,dy$$

for $x \in B(x_0, N)$.

Using the size condition on K, we see that the above is

$$\le \int_{|x-y|>N} C_K N^{-d}\chi_{B(x_0,2N)}(y)\,dy \lesssim_K N^{-d}(c_d(2N)^d) \lesssim_d 1,$$

and so

$$I_{\epsilon,r_B/2}(x) = (T_{\epsilon,r_B/2}(\chi_B))(x) = (T_\epsilon(\chi_B))(x) + O_{d,K}(1)$$

for all balls $B \subseteq \mathbb{R}^d$ with $x \in \frac{1}{2}B$, $r_B > 2\epsilon$.

As a consequence, let $\epsilon > 0$, $R > 0$, with $\epsilon < R$. Take any ball $B \subseteq \mathbb{R}^d$; then

$$(T_{\epsilon,R}(\chi_B))(x) = (T_\epsilon(\chi_B))(x) - \int_{|x-y|>R} K(x,y)\chi_B(y)\,dy,$$

and by a similar calculation, we have that the second term is bounded by $\lesssim_{d,K} R^{-d}\lambda^d(B)$. So if we have a small ball, say, $\mathrm{diam}(B) \le 1000R$, we see that

$$\int_{\frac{1}{2}B} |T_{\epsilon,R}(\chi_B)|\,d\lambda^d \le O_{d,K}(\lambda^d(B)) + \int_{\frac{1}{2}B} |T_\epsilon(\chi_B)|\,d\lambda^d,$$

and we know that if $r_B > 2\epsilon$, then this can be further estimated by

$$\leq O_{d,K}(\lambda^d(B)) + \int_{\frac{1}{2}B} |I_{\epsilon,r_B/2}(x)|\,dx + O_{d,K}(\lambda^d(B)),$$

and thus the entire expression is controlled by $\leq (C_{d,K} + C)\lambda^d(B)$.

For balls with $r_B \leq 2\epsilon$, we note that

$$(T_{\epsilon,R}(\chi_B))(x) = \int_{\epsilon<|x-y|<R} K(x,y)\chi_B(y)\,dy,$$

which is controlled in absolute value by

$$\leq C_K\epsilon^{-d}\lambda^d(B) \lesssim_{d,K} 1.$$

As a consequence, we need only deal with the very large balls now, $B \subseteq \mathbb{R}^d$ with $\operatorname{diam}(B) \geq 1000R$ and $r_B \geq 2\epsilon$.

To do this, we take the ball $\frac{1}{2}B$, and observe that we can cover it by $\lesssim_d (r_B/R)^d$ balls of radius R. Indeed, we embed the ball $\frac{1}{2}B$ within a cube, which will have side-length $\sim_d r_B$, and round up its side-length to the nearest power of two (which will preserve this relation). For any $x \in B$, we likewise extract the largest cube of dyadic-integer side-length within $B(x,R)$ (which will be comparable to R, up to a dimensional constant), and translate to create a dyadic system of cubes centered at $c_B \in \mathbb{R}^d$. These smaller dyadic cubes will then fit exactly, and we can cover $\frac{1}{2}B$, using $\lesssim_d r_B^d/R^d$ such cubes. We finally extract only those cubes whose associated R-radius spheres meet $\frac{1}{2}B$; we will cover $\frac{1}{2}B$ with only these spheres. They will produce a neighborhood, all contained in $(\frac{1}{2}+\epsilon)B$.

Now, we cover $\frac{1}{2}B$ by this family, and write

$$\int_{\frac{1}{2}B} |(T_{\epsilon,R}(\chi_B))(x)|\,dx \leq \sum_i \int_{B(x_i,R)} |(T_{\epsilon,R}(\chi_B))(x)|\,dx$$

and we note that

$$(T_{\epsilon,R}(\chi_B))(x) = \int_{\epsilon<|x-y|<R} K(x,y)\chi_B(y)\,dy = (T_{\epsilon,R}(\chi_{B(x_i,2R)}))(x)$$

for $x \in B(x_i,R)$, since $|y-x_i| \leq |y-x| + |x-x_i| < R+R$ and also $B(x_i,2R) \subseteq (\frac{1}{2}+\epsilon)B \subseteq B$ by the size condition on R relative to B.

We may now use that

$$(T_{\epsilon,R}(\chi_{B(x_0,2R)}))(x) = I_{\epsilon,R}(x) + O_{d,K}(1)$$

for $x \in B(x_0,R)$, provided $R > 2\epsilon$. (If $R \leq 2\epsilon$, we use the same argument with the size condition to bound the left-hand side purely by $O_{d,K}(1)$.)

Now, in either case, integrating this, we obtain that

$$\int_{B(x_i,R)} |(T_{\epsilon,R}(\chi_{B(x_i,2R)}))(x)|\, dx \leq CR^d + O_{d,K}(R^d).$$

Summing over indices i and using the cardinality estimate, we can then bound

$$\int_{\frac{1}{2}B} |T_{\epsilon,R}(\chi_B)|\, d\lambda^d \lesssim_d (r_B^d/R^d)(C + C_{d,K})R^d \lesssim_{d,K,I} r_B^d.$$

As a consequence, we have

$$\int_B |T_{\epsilon,R}(\chi_{2B})|\, d\lambda^d \lesssim_{d,K,I} \lambda^d(B)$$

for all $B \subseteq \mathbb{R}^d$, uniformly in ϵ and R. Subtracting

$$(T_{\epsilon,R}(\chi_{2B} - \chi_B))(x) = \int_{\epsilon<|x-y|<R} K(x,y)\chi_{(2B)\setminus B}(y)\, dy,$$

and recalling that we showed

$$|(T_{\epsilon,R}(\chi_{2B} - \chi_B))(x)| \leq \int_{(2B)\setminus B} C_K|x-y|^{-d}\, dy \lesssim_{d,K} 1 + \log_2\left(\frac{r_B}{d(x,\partial B)}\right)$$

for $x \in B$, in the argument for the $T(1)$ theorem (by using the size condition on K), we have that

$$\int_B |T_{\epsilon,R}(\chi_{2B} - \chi_B)|\, d\lambda^d \lesssim_{d,K} \lambda^d(B),$$

and uniformly over balls $B \subseteq \mathbb{R}^d$. It follows that

$$\int_B |T_{\epsilon,R}(\chi_B)|\, d\lambda^d \lesssim_{d,K,I} \lambda^d(B).$$

By the boundedness of the $I^*_{\epsilon,R}$ as well, repeating the arguments above *mutatis mutandis*, we get a similar uniform bound (over all balls $B \subseteq \mathbb{R}^d$) for the adjoint operators $T^*_{\epsilon,R}$. So by the corollary to Tao's $T(1)$ theorem, we get

$$\|T_{\epsilon,R}\|_{L^2\to L^2} \lesssim_{d,\delta,K} 1,$$

for all $\epsilon > 0$, $R > 0$, $\epsilon < R$, where the $T_{\epsilon,R}$ are the smooth truncations to use the $T(1)$ theorem. (We require operators T with kernels that obey the smoothness condition, but for the sharp-truncations $T_{\epsilon,R}$, we may add the spillover contributions from smoothly-truncated kernel, at length scales ϵ and R ($R > 2\epsilon$), to get an error term consisting of integrals over $\epsilon < |\cdot| < 2\epsilon$ and $R < |\cdot| < 2R$. Upon integrating χ_B over these sets against the kernel, using the size condition, this produces an excess of $O_d(1)$ pointwise, which still integrates to $O_d(\lambda^d(B))$. Meanwhile, $\epsilon < R < 2\epsilon$ follows from the size condition and the Hardy-Littlewood maximal function, and we need not consider any of these tools.)

Then as $\epsilon \downarrow 0$ and $R \uparrow \infty$, there will exists a subsequence of (ϵ_j, R_j) with a weak limit $T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ to which the T_{ϵ_j,R_j} converge, and this will be a Calderón-Zygmund operator with kernel K.

Consequently, if K satisfies this cancellation condition, we can say there does exist a Calderón-Zygmund operator T with associated kernel K. $\square$

Remark: By Cotlar's inequality for standard Calderón-Zygmund operators, we see that if T is L^2-bounded and is represented by a standard kernel K, then it follows that T_ϵ is, also (for any $\epsilon > 0$). Then writing $T_{\epsilon,R} = T_\epsilon - T_R$, we get that the $T_{\epsilon,R}$ are also L^2-bounded, uniformly in ϵ and R. Then the bounds on the $I_{\epsilon,R}$ and $I^*_{\epsilon,R}$, from the hypotheses, are consequences of the L^2-boundedness of those operators, which will in turn be implied by the L^2-boundedness of T. So we can see that the condition is *necessary* for a kernel to be associated with some Calderón-Zygmund operator, as well as sufficient.

References

[1] Elias M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, N.J., 1993. MR 1232192.

[2] Terence Tao, *Lecture Notes 4 for 247A*, MATH 247A: Fourier Analysis, UCLA Mathematics (Nov. 2006), `https://www.math.ucla.edu/~tao/247a.1.06f/notes4.pdf`.

[3] Terence Tao, *Lecture Notes 7 for 247B*, MATH 247B: Fourier Analysis, UCLA Mathematics (Feb. 2007), `https://www.math.ucla.edu/~tao/247b.1.07w/notes7.pdf`.

Appendix

Theorem: Let T be a Calderón-Zygmund operator whose off-support representation has vanishing kernel. Then T is a spatial multiplier operator.

Proof. We define the set function $\mu(E) = \langle T(\chi_E), \chi_E \rangle$, where $\langle \cdot, \cdot \rangle$ is the standard $L^2(\mathbb{R}^d)$ pairing (linear or antilinear; it does not matter for this).

We then see that μ is a countably-additive set function; indeed, if E and F are disjoint, then $T(\chi_E)\chi_F = 0$, and *vice versa*, and thus

$$\langle T(\chi_E + \chi_F), \chi_E + \chi_F \rangle = \langle T(\chi_E), \chi_E \rangle + \langle T(\chi_F), \chi_F \rangle,$$

and the left-hand side is equal to $\langle T(\chi_{E \cup F}), \chi_{E \cup F} \rangle = \mu(E \cup F)$.

By induction, this holds for finite collections of disjoint finite-measure sets, and by continuity on $L^2(\mathbb{R}^d)$, we get countable-additivity, taking the underlying measure space to be equal to any ball $B(0, R)$ for large R.

By Cauchy-Schwarz, and the L^2-boundedness of the Calderón-Zygmund operator, we see that $|\langle T(\chi_E), \chi_E \rangle| \lesssim_T \lambda^d(E)$.

It follows that $\mu \mid_{B(0,R)}$ is a complex measure, absolutely continuous with respect to $\lambda^d \mid_{B(0,R)}$, and so by the Radon-Nikodym theorem, there exists $b_R : B(0, R) \to \mathbb{C}$ which represents $\mu \mid_{B(0,R)}$:

$$\mu(E) = \langle T(\chi_E), \chi_E \rangle = \int_E b_R \, d\lambda^d$$

for Borel sets $E \subseteq B(0, R)$.

Moreover, by the inequality above, we have

$$\|b_R\|_{L^\infty(B(0,R))} \leq \|T\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)}.$$

We can also see that there must be consistency: taking larger-and-larger values of R, we note that the functions b_R are simply restrictions (to their respective domains $B(0, R)$) of a single function $b \in L^\infty(\mathbb{R}^d)$, obeying the same L^∞ bound.

Since $T(\chi_E) = 0$ a.e. outside E, it follows that the above $\mu(E)$ is precisely

$$\int_{\mathbb{R}^d} T(\chi_E) \, d\lambda^d = \int_{\mathbb{R}^d} \chi_E b \, d\lambda^d,$$

for bounded Borel sets $E \subseteq \mathbb{R}^d$. Now, for any $F \subseteq \mathbb{R}^d$, we claim that

$$T(\chi_E)\chi_F = T(\chi_E \chi_F).$$

Indeed, we see that

$$T(\chi_E)\chi_F = T(\chi_E)\chi_E\chi_F = T(\chi_E)\chi_{E \cap F} = (T(\chi_{E \cap F}) + T(\chi_{E \setminus F}))\chi_{E \cap F}$$

by linearity of the operator T, and the second term is zero, as $T(f)$ vanishes at a point whenever f vanishes in a neighborhood thereof.

(If the Calderón-Zygmund condition is initially stated for $f \in C_c^\infty(\mathbb{R}^d)$, we may take limits, in a Urysohn-type approximation: we first have that $T(f) = 0$ in K^c when $\{f \neq 0\} \subseteq K$, K compact; then, using inner and outer regularity, this is true for all indicator functions of bounded Borel sets E, and we may take further limits to move from simple functions to all L^2-ones. Then to get vanishing a.e. of $T(\chi_A)\chi_B$ for $A \cap B = \emptyset$, A, B of finite measure, we note that $\langle |T(\chi_C)|, \chi_K \rangle = 0$ for all $C \subseteq A$, $K \subseteq B$ compact, and approximate.)

Hence, we may take

$$\int_{\mathbb{R}^d} T(\chi_E)\chi_F \, d\lambda^d = \int_{\mathbb{R}^d} \chi_E \chi_F b \, d\lambda^d,$$

for all bounded Borel sets $E, F \subseteq \mathbb{R}^d$.

Taking linear combinations of various disjoint Borel sets E, all supported within a single large compact region, we can see that

$$\int_{\mathbb{R}^d} T(\chi_E) g \, d\lambda^d = \int_{\mathbb{R}^d} (\chi_E b) g \, d\lambda^d$$

for any bounded g, vanishing outside a compact set; we may take an approximate identity. Then $T = M_b$ on simple functions, and hence, all functions. $\square$